Averaged Lagrangian

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Lagrangian formulation

Field equation, e.g. Klein-Gordon

$$\frac{\partial}{\partial t}\left(\alpha\frac{\partial\phi}{\partial t}\right) = \frac{\partial}{\partial x}\left(\beta\frac{\partial\phi}{\partial x}\right) - \gamma\phi,$$

with α, β & γ functions of slow X & T,

has a Lagrangian formlation $\delta L = 0$ with

$$L = \int \mathcal{L}\, dxdt,$$

using a Lagrangian density $\mathcal{L}$, e.g. for our KG

$$\mathcal{L} = \tfrac{1}{2}\alpha\phi_t^2 - \tfrac{1}{2}\beta\phi_x^2 - \tfrac{1}{2}\gamma\phi^2.$$

Euler-Lagrange equation

Derive the field equation as the Euler-Lagrange equation

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\phi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\phi_x}\right) + \cdots = 0.$$

More generally,

$$\mathcal{L} = \mathcal{L}\left(\phi, \phi_t, \phi_x, \phi_{xx}; X, T\right).$$

Slowly varying wave

Now substitute a slowly varying wave

$$\phi = a(X,T)\cos\theta, \quad \theta = \frac{1}{\epsilon}\Theta$$

$$\text{with} \begin{cases} \text{local wavenumber} & k = \theta_x = \Theta_X \\ \text{local frequency} & \omega = -\theta_t = -\Theta_T. \end{cases}$$

Hence

$$\mathcal{L} = \mathcal{L}\left(a\cos\theta, a\omega\sin\theta, -ak\sin\theta, -ak^2\cos\theta, \ldots; X, T\right),$$

changing to generalised coordinates a and θ.

Averaging

When evaluating L, first integrate over *fast* θ

$$\overline{\mathcal{L}} = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(a, \theta; X, T)\, d\theta$$

and then integrate over *slow* X, T

$$L = \frac{1}{\epsilon^2}\int \overline{\mathcal{L}}\, dXdT$$

E.g. KG equation

$$\overline{\mathcal{L}} = \tfrac{1}{4}\alpha\omega^2 a^2 - \tfrac{1}{4}\beta k^2 a^2 - \tfrac{1}{4}\gamma a^2.$$

Governing equations

Euler-Lagrange equations for variations δa and $\delta\theta$.

$$\delta a: \qquad \frac{\partial\overline{\mathcal{L}}}{\partial a} = 0 \quad \text{gives dispersion relation}$$

For KG

$$\alpha\omega^2 = \beta k^2 + \gamma.$$

Equipartition

In general linear waves

$$\overline{\mathcal{L}} = \tfrac{1}{4}a^2 F(\omega, k)$$

with Euler-Lagrange equation

$$\delta a: \qquad \text{dispersion relation} \quad F(\omega, k) = 0.$$

But

$$\overline{\mathcal{L}} = \overline{T} - \overline{V},$$

so $F = 0$ is the equipartition of energy.

Variation of phase $\delta\theta$

Now θ only occurs in $\overline{\mathcal{L}}$ as slow variables $\theta_t = -\omega$ & $\theta_x = k$.

$$\delta\theta: \qquad 0 - \frac{\partial}{\partial T}\left(-\frac{\partial\overline{\mathcal{L}}}{\partial\omega}\right) - \frac{\partial}{\partial X}\left(\frac{\partial\overline{\mathcal{L}}}{\partial k}\right) = 0,$$

which is in *conservation* form with density $\frac{\partial\overline{\mathcal{L}}}{\partial\omega}$ and flux $\frac{\partial\overline{\mathcal{L}}}{\partial k}$.

For linear waves $\overline{\mathcal{L}} = \frac{1}{4}a^2 F(\omega, k)$, so

$$\frac{\partial\overline{\mathcal{L}}}{\partial k} = \tfrac{1}{4}a^2\frac{\partial F}{\partial k} = -\tfrac{1}{4}a^2\frac{\partial F}{\partial\omega}\left(\frac{\partial\omega}{\partial k}\right)_{F=0} = -c_g\frac{\partial\overline{\mathcal{L}}}{\partial\omega}.$$

Hence conservation of wave action $A = \frac{\partial\overline{\mathcal{L}}}{\partial\omega}$

$$\frac{\partial A}{\partial T} + \frac{\partial}{\partial X}(c_g A) = 0$$