Mathematical Tripos Part III E.J.Hinch
Slow Viscous Flow February 2008

Example Sheet 1

All flows after Q1 are Stokes flows. The body force $\mathbf{f}$ *is zero unless stated otherwise.*

1. A rigid sphere of radius a oscillates in fluid of kinematic viscosity ν with the displacement of the centre given by $x = A\cos\omega t$. Under what conditions on a, A, ω and ν can the motion be approximately described by (i) the Stokes equations (ii) the unsteady Stokes equations?

2. Two identical rigid spheres settle under gravity through an unbounded region of very viscous fluid. What does reversibility imply about their velocities and angular velocities?

3. A homogeneous ellipsoid falls under gravity in an unbounded viscous fluid. Show that it, and any body with three orthogonal planes of symmetry, does not rotate.

4. Verify by direct substitution that the Stokes equations are satisfied by the expression

$$\mathbf{u} = \mathbf{E}\cdot\mathbf{x} + \nabla(\mathbf{x}\cdot\boldsymbol{\Phi}) - 2\boldsymbol{\Phi} \quad \text{with} \quad p = 2\mu\nabla\cdot\boldsymbol{\Phi},$$

where $\boldsymbol{\Phi}$ is a harmonic vector function and $\mathbf{E}$ is a constant traceless second-rank tensor. Show further that if $\mathbf{E}$ is symmetric then the stress tensor is given by

$$\boldsymbol{\sigma} = 2\mu\left[\mathbf{E} - (\nabla\cdot\boldsymbol{\Phi})\mathbf{I} + (\nabla\nabla\boldsymbol{\Phi})\cdot\mathbf{x}\right].$$

Hence find the velocity and stress fields for a point source of volume flux Q (in otherwise stationary fluid).

5. Use the Papkovich-Neuber representation of Stokes flow to derive the flow $\mathbf{u}$ due to a couple $\mathbf{G}$ acting on a rigid sphere, radius a centred at $\mathbf{x} = \mathbf{0}$, in an unbounded fluid.

6. Determine the flow outside a rigid sphere rotating with prescribed angular velocity $\boldsymbol{\omega}$ in an applied linear flow, i.e. the flow that has

$\mathbf{u} = \boldsymbol{\omega}\wedge\mathbf{x}$ on $r = a$, $\quad \mathbf{u} \to \boldsymbol{\Omega}\wedge\mathbf{x} + \mathbf{E}\cdot\mathbf{x}$ as $r \to \infty$ with $\mathbf{E}$ symmetric & traceless

[*Hint:* Use linearity to decompose the problem.] If the sphere is now allowed to rotate freely, what is $\boldsymbol{\omega}$?

7. Find the flow inside and outside a spherical droplet of radius a, viscosity $\lambda\mu$ and density ρ_1 translating at fixed velocity $\mathbf{V}$ through a fluid of viscosity μ and density ρ_2. [*Hints:* Guess suitable potentials $\boldsymbol{\Phi}$ for the interior and exterior flows and, for the moment, ignore the jump condition on $\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}$. $\mathbf{V}$ is the velocity *of* the boundary not the fluid velocity *on* the boundary. Make the problem dimensionless.] Show that the drag on the droplet is

$$\mathbf{F} = -4\pi\mu a\mathbf{V}\left(\frac{1 + \frac{3}{2}\lambda}{1 + \lambda}\right)$$

Hence determine the speed of (i) a solid sphere and (ii) a spherical bubble in a gravitational field. *What balances the apparent jump in $\mathbf{n}\cdot\boldsymbol{\sigma}\cdot\mathbf{n}$?*

8. For a particle translating through an unbounded viscous fluid, show that the drag force (that part of the force parallel to its velocity) is higher when the Reynolds number is not zero compared with the Stokes drag.

9. Newtonian fluid of viscosity μ fills the gap between two long concentric circular cylinders of radii R_1 and $R_2\,(>R_1)$; the inner cylinder is stationary and the outer is made to rotate steadily with angular velocity Ω. Find the azimuthal velocity profile $u_\theta(r)$ in $R_1 \le r \le R_2$. [*Hint:* Use P–N potentials or note that the couples on any cylindrical fluid shell balance.] Hence determine the couple (torque) per unit length applied to the outer cylinder.

Suppose now that a number of rigid force-free couple-free particles are suspended in the fluid, and the same torque as before is applied to the cylinders. Show that the rate of rotation is decreased, explaining your argument carefully. [*Hint:* You will need to string together a number of results.]

10. A rigid particle moves through a region of very viscous fluid enclosed by a stationary rigid boundary. Show that the rate of dissipation is given by $D = \mathbf{F}\cdot\mathbf{U}+\mathbf{G}\cdot\boldsymbol{\Omega}$, where $\mathbf{F}$ and $\mathbf{G}$ are the total force and couple on the particle, and $\mathbf{U}$ and $\boldsymbol{\Omega}$ are its velocity and angular velocity.

Use the reciprocal theorem to deduce the grand resistance matrix in

$$\begin{pmatrix}\mathbf{F}\\ \mathbf{G}\end{pmatrix} = \begin{pmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{pmatrix}\begin{pmatrix}\mathbf{U}\\ \boldsymbol{\Omega}\end{pmatrix}$$

is symmetric, i.e. $\mathbf{A} = \mathbf{A}^T$, $\mathbf{B} = \mathbf{C}^T$ and $\mathbf{D} = \mathbf{D}^T$.

Now show that for a cube $\mathbf{A}$ and $\mathbf{D}$ are diagonal, while $\mathbf{B}$ and $\mathbf{C}$ vanish.

11. An axisymmetric body falls slowly under gravity through viscous incompressible fluid. The shape is such that the motion is one of pure translation. In one orientation the velocity $\mathbf{V}$ of the body makes an angle $\theta = \theta_1$ with the symmetry axis, and an angle $\alpha = \alpha_1$ with the downward vertical; in another the body falls at a different velocity $\mathbf{U}$, with $\theta = \alpha = \phi$, say. Prove that

$$\tan^2\phi = 1 - 2\tan\theta_1\cot(\theta_1+\alpha_1)$$

and that

$$\mid \mathbf{U} \mid = \mid \mathbf{V} \mid \left[\frac{2\sin\theta_1\cos\phi}{\sin(\theta_1+\alpha_1)}\right].$$

12. Determine the Faxen formula

$$\boldsymbol{\Omega} = \tfrac{1}{2}\boldsymbol{\omega}_\infty(x_0) + \frac{\mathbf{G}}{8\pi a^3\mu}$$

for the rate of rotation of a rigid sphere placed in an arbitrary flow $\mathbf{u}_\infty(\mathbf{x})$ and which has a couple $\mathbf{G}$ applied to it.

13. Apply the reciprocal theorem to the flow $\mathbf{u}_1(\mathbf{x})$ due to a body force $\mathbf{f}(\mathbf{x})$ outside a force-free sphere and the flow $\mathbf{u}_2(\mathbf{x})$ due to a force $\mathbf{F}$ acting on a sphere in a fluid with no body force acting on it. Deduce that the force-free sphere moves at a velocity

$$\mathbf{V} = \frac{1}{8\pi\mu}\int_{r>a}\left[\mathbf{f}(\mathbf{x})\left(\frac{1}{r}+\frac{1}{3}\frac{a^2}{r^3}\right) + \mathbf{x}(\mathbf{f}(\mathbf{x})\cdot\mathbf{x})\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\right]dV$$

[*Note: 'Force-free' means* $\int \boldsymbol{\sigma}\cdot\mathbf{n}\,dS = \mathbf{0}$ *not* $\boldsymbol{\sigma}\cdot\mathbf{n} = \mathbf{0}$*!*]