Chapter 10 – Strong flows

- Birefreingent strand
  - thin layer of high stress leaqving a stagnation point
- Wine-glass model of contraction flow
  - anisotropic flow from anisotropic material
- Corner singularity
  - fast flow with no relaxation
- Limited-forec flows
  - strain only to avoid relaxation

Oldroyd-B, and its limitations

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{1}{\tau}(A - \mathbf{I})$$
$$\sigma = -p\mathbf{I} + 2\mu_0 E + G f A$$

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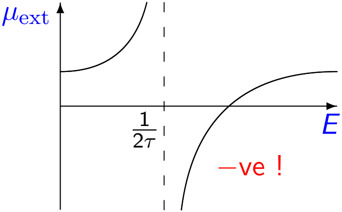
Steady extensional flow



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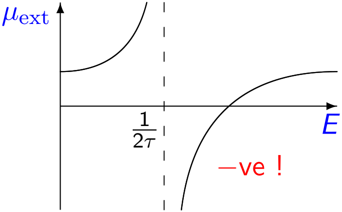

Microstructure deforms without limit if $E > \frac{1}{2\tau}$: $\quad A = e^{(2E - \frac{1}{\tau})t}$

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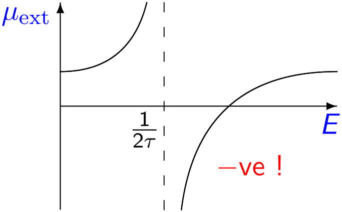

Microstructure deforms without limit if $E > \frac{1}{2\tau}$: $\quad A = e^{(2E - \frac{1}{\tau})t}$

Need to limit deformation of microstructure

FENE modification

Finite Extension Nonlinear Elasticity

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{f}{\tau}(A - \mathbf{I})$$

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$$f = \frac{L^2}{L^2 - \mathrm{trace}\, A} \qquad \text{keeps} \quad A < L^2$$

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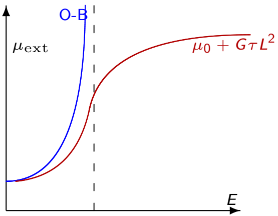

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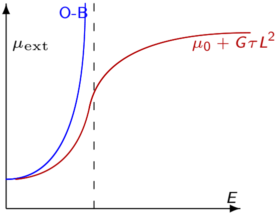


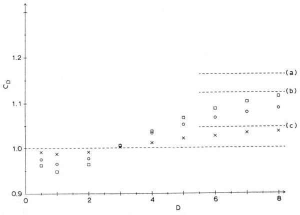
Will use FENE, and if safe Oldroyd-B, in following strong flows

FENE flow past a sphere

Oldroyd-B gave decrease is drag

FENE

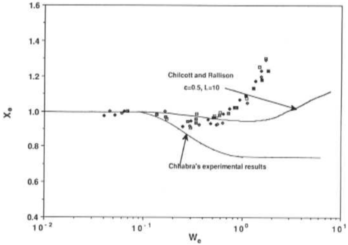
F



De

Chilcott & Rallison 1988 JNNFM

Experiments M1



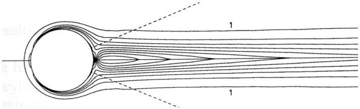
De

Tirtaatmadja, Uhlherr & Sridhar 1990 JNNFM

FENE gives drag increase

. . . FENE flow past sphere

FENE drag increase from long wake of high stress



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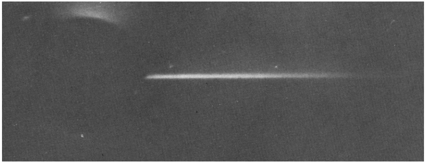
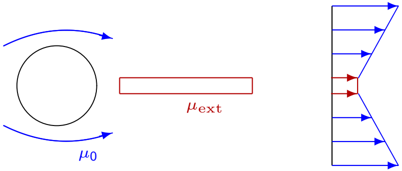


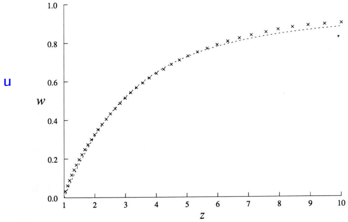
Cressely & Hocquart 1980 Opt Act

“Birefringent strand”

...birefringent strands

Boundary layers of high stress.
Crude model: μ_{ext} in wake, μ_0 elsewhere.

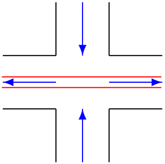


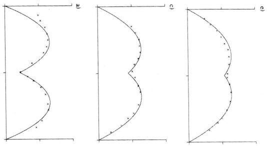


Harlen, Rallison & Chilcott 1990 JNNFM

. . . birefringent strands

Can apply to all flows with stagnation points, e.g.

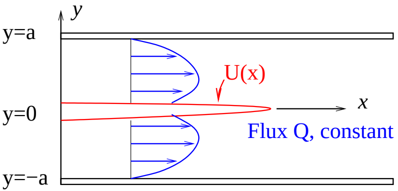




Harlen, Rallison & Chilcott 1990 JNNFM

Also cusps at rear stagnation point of bubbles.

Analysis of birefringent strand in exit channel



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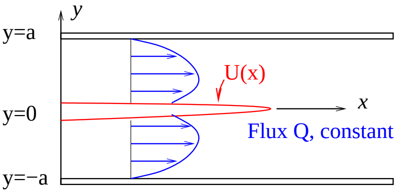

Lubrication flow

$$u(x, y) = U(x)\frac{a-y}{a} + (Q - Ua)\,\frac{3y(a-y)}{a^2}$$

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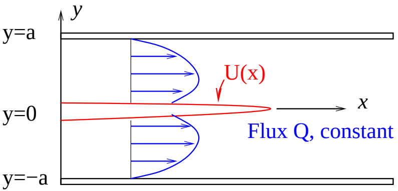

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Force balance on strand

$$\left[\mu\frac{\partial u}{\partial y}\right]_{0-}^{0+} + \frac{\partial}{\partial x}\left(\delta\mu_{\mathrm{ext}}\frac{\partial U}{\partial x}\right)$$

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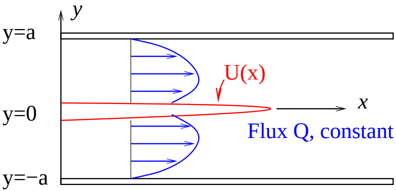

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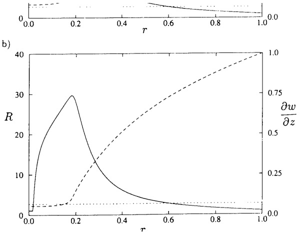
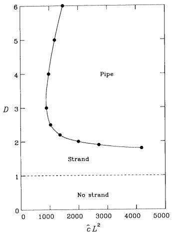
Solving (Student Exercise)

$$U(x) = \frac{3Q}{2a}\left(1 - e^{-\sqrt{\frac{8\mu}{\delta\mu_{\text{ext}}a}}x}\right)$$

Birefringent pipes

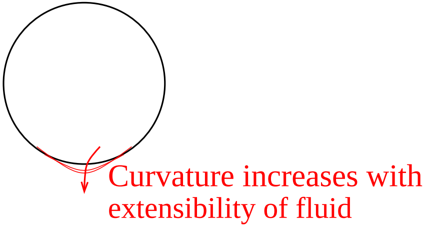
Very low extension rate in the strand can fail to stretch the microstruture, so relax, producing birefringent "pipes".





Harlen, H, Rallison (1992) JNNFM 44

Formation of a cusp at rear stagnation point of a bubble



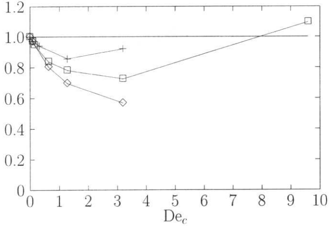
Rallsion & Malaga (2007) JNNFM 141

FENE contraction flow

Oldroyd-B gave decrease is pressure drop

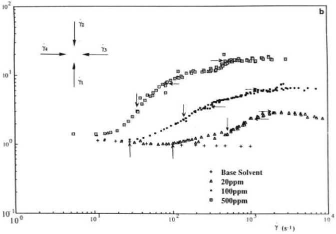
FENE $L = 5$

Δp



Szabo, Rallison & Hinch 1997 JNNFM

Experiments



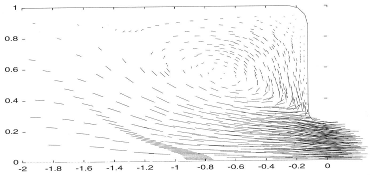
Cartalos & Piau 1992 JNNFM

FENE gives increase in pressure drop

...FENE contraction flow

Increase in pressure drop from long upstream vortex

FENE $L = 5$



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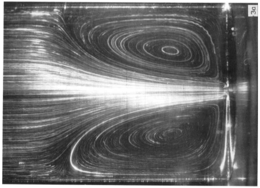

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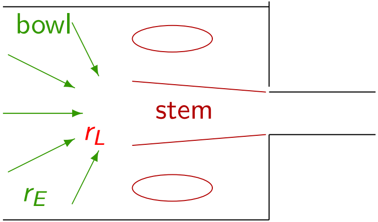
. . . a champagne-glass model



Bowl:

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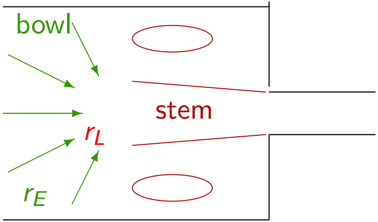

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- Sink flow $u = \frac{Q}{2\pi r^2}$

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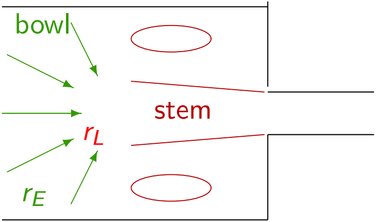

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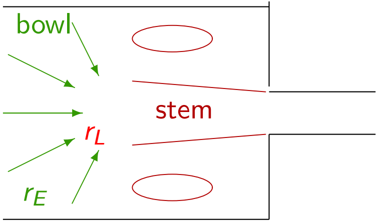

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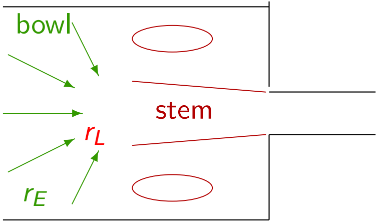

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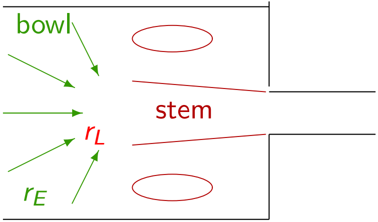

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- Hence fully stretched only if $De = \frac{Q\tau}{d^3} > L^{3/2}$.

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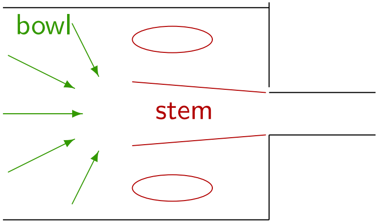

Stem:

- Fully stretched, $A \approx L^2$,

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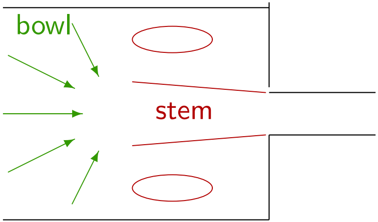

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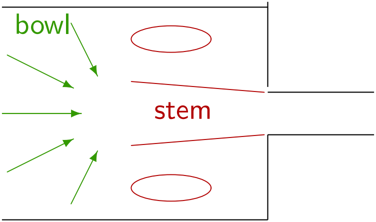

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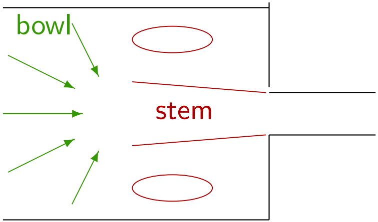

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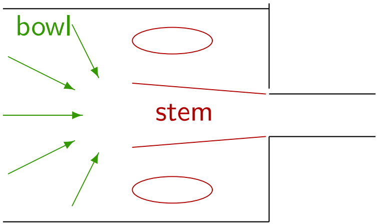

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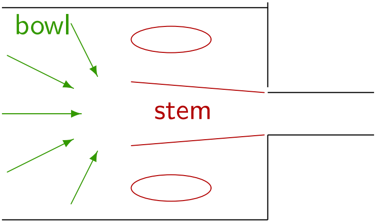
...a champagne-glass model



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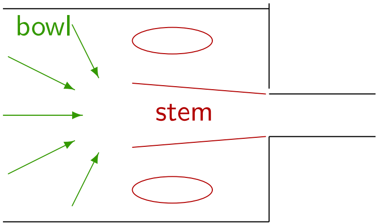
Szabo, Rallison & H (1997) JNNFM 72

...a champagne-glass model



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Szabo, Rallison & H (1997) JNNFM 72

Flow anisotropy from material anisotropy: $\mu_{\text{ext}} \gg \mu_{\text{shear}}$ TDR

Fast flows with no relaxation

If $\nabla \mathbf{u} \gg \frac{1}{\tau}$

$$\frac{DA}{Dt} = A \cdot \nabla \mathbf{u} + \nabla \mathbf{u}^T \cdot A - \frac{1}{\tau} \left(A - \mathbf{I} \right)$$

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Recall material line elements

$$\frac{d}{dt}\delta\ell = \delta\ell \cdot \nabla \mathbf{u},$$

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Suggests steady solution $\quad$ ($g(\psi)$ from matching to slower region)

$$A = g(\psi)\mathbf{uu}, \quad \text{so } \sigma = -p\mathbf{I} + 2\mu_0 E + Gg\mathbf{uu}$$

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Tensions in streamlines again

Fast flows with no relaxation 2

Momemtum, ignoring viscous stress

$$0 = -\nabla p + G g^{1/2} \mathbf{u} \cdot \nabla g^{1/2} \mathbf{u}.$$

Euler equation!!

Fast flows with no relaxation 2

Momemtum, ignoring viscous stress

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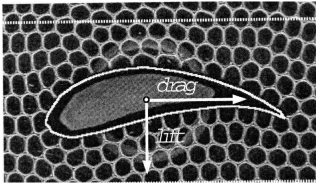
Euler equation!!

Anti-Bernoulli

$$p - \frac{1}{2} G g u^2 = \text{const}$$



Dollet, Aubouy & Graner 2005 PRL

Fast flows with no relaxation 3

Potential flows $g^{1/2}\mathbf{u} = \nabla\phi$

Fast flows with no relaxation 3

Potential flows $g^{1/2}\mathbf{u} = \nabla\phi$

Flow around sharp 270° corner: Hinch 1995 JNNFM

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Fast flows with no relaxation 3

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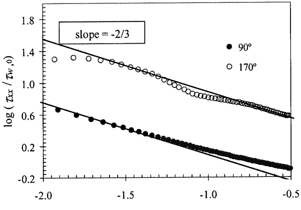

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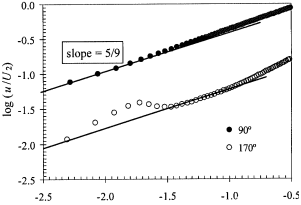

(a)

Alves, Oliviera & Pinho 2003 JNNFM

Fast flows with no relaxation 4

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Details of the boundary layers – very difficult

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Oldroyd-B becomes Student Exercise

$$\begin{aligned}
\mathbf{u} \cdot \nabla \lambda &= \frac{2\gamma}{u^2}\mu - \frac{1}{\tau}\left(\lambda - \frac{1}{u^2}\right) \\
\mathbf{u} \cdot \nabla \mu &= \frac{\gamma}{u^2}\nu - \frac{1}{\tau}\mu \\
\mathbf{u} \cdot \nabla \nu &= -\frac{1}{\tau}\left(\nu - u^2\right)
\end{aligned}$$

with

$$\gamma = \mathbf{v} \cdot (\nabla \mathbf{u} + \nabla \mathbf{u}^T) \cdot \mathbf{u} = -u^2 \nabla \cdot \mathbf{v}$$

Renardy (1994) JNNFM 52

Capillary squeezing – controlled by relaxation

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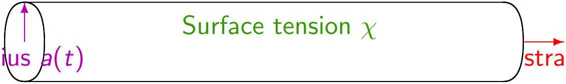

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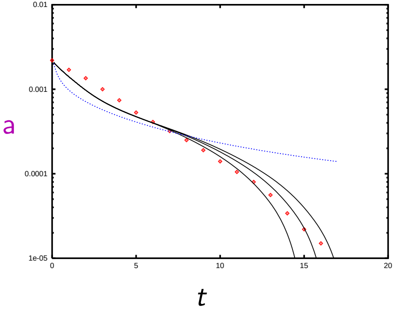
Need slow $E = 1/3\tau$ to stop A_{zz} relaxing from $\chi / G a$

. . . capillary squeezing

Oldroyd-B $\quad a(t) = a(0)e^{-t/3\tau} \quad$ does not break

Experiments S1 fluid



Exp: Liang & Mackley 1994 JNNFM

Thy: Entov & Hinch 1997 JNNFM

but filament eventually breaks in experiments

Multi-mode generalisation

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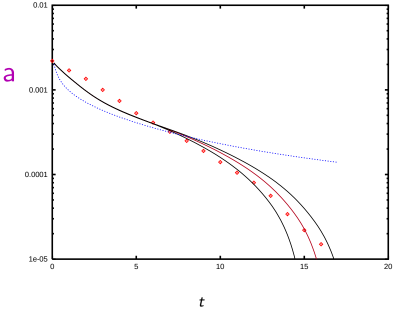
Spectrum needed to fit experiments at middle times

FENE capillary squeezing

Filament breaks in with FENE $L = 20$



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