Chapter 3

Chapter 1: the phenomena.

Chapter 2: measuring intrinsic properties, e.g. viscosity and elasticity.

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Chapter 2: measuring intrinsic properties, e.g. viscosity and elasticity.

Now need to encapsulate those properties in governing equations.

Conservation equations – true all materials

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- Need Constitutive (material dependent) Relation between stress σ and flow $\mathbf{u}$.

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‘Simple’ materials

Lagrangian description

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Deformation of line element (for micro-lengths $\ll$ macro-lengths)

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This functional dependence not useful, except for fast elastic limit and slow viscous limits (each with single parameter)

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Require $\sigma\{A\}$ to obey this identity for all $Q(t)$.

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In incompressible material with isotropic in rest state,

$$f(U^2) = U^2 f_1 + U^{-2} f_2, \quad f_i \text{ scalar functions of invariants of } U,$$

so

$$\sigma = AA^T f_1 + A^{-1\,T} A^{-1} f_2$$

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Virtual work and σ co-diagonal with U gives

$$\sigma_1 = \frac{1}{\lambda_2\lambda_3}\left(\frac{\partial w}{\partial \lambda_1} = \lambda_1 \frac{\partial w}{\partial \alpha} - \lambda_1^{-3}\frac{\partial w}{\partial \beta}\right),$$

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Better for data fitting – Ogden model: $w(\lambda_1^n + \lambda_2^n + \lambda_3^n)$

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strain-rate $E' = QEQ^T$, vorticity (tensor) $\Omega' = Q\Omega Q^T - \dot{Q}Q^T$

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This is the rate of change of σ seen by an observer rotating with the vorticity, and so is universal.

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Recall rotation frames

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so for second-order tensor

$$\dot{\delta\ell\delta\ell} = \nabla u^T \cdot \delta\ell\delta\ell + \delta\ell\delta\ell \cdot \nabla u$$

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For a Newtonian viscous fluid $G(s) = \delta(s)$ and for an elastic solid $G(s) = 1$.

Linear viscoelasticity 2

Student Exercise: If $G(t)$ has a single exponential decay,

$$G(t) = G_0 e^{-t/\tau}$$

show that a polar plot of $Re(G^*)$ versus $Im(G^*)$ as (real) ω varies is part of a circle.

Linear viscoelasticity 3

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Hence recoil after stop steady shear flow $\dot{\gamma}_0$

$$-\dot{\gamma}_0 \frac{\int_0^\infty sG(s)\,ds}{\int_0^\infty G(s)\,ds}$$

Student Exercise

Second-order fluid

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Dangerous in stability analyses and numerical studies, where bad behaviour can occur outside limitation of weak and slowly varying.

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Student Exercises

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In simple shear

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- constant viscosity μ

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- Normal stress difference $N_1 = 2\alpha\dot{\gamma}^2$, $N_2 = -\frac{1}{4}\beta\dot{\gamma}^2$

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In (axisymmetric pure) extensional flow

- $\mu_{\mathrm{ext}} = \mu + \left(\alpha + \frac{1}{4}\beta\right)\dot{\epsilon}$

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- but must keep last term small

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- Carreau, Yasuda & Cross

$$\mu = \mu_\infty + (\mu_0 - \mu_\infty)\left(1 + (\tau\dot{\gamma})^a\right)^{(n-1)/a}$$

with plateaux at high and low $\dot{\gamma}$.

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- constant viscosity $\mu = \mu_0$
- Normal stress difference $N_1 = 2\mu_0(\lambda_1 - \lambda_2)\dot{\gamma}^2$, $N_2 = 0$

Oldroyd-B model fluid 2

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- $$\mu_{\text{ext}} = \mu_0 \frac{1 - \lambda_2\dot{\epsilon} - 2\lambda_1\lambda_2\dot{\epsilon}^2}{(1 - 2\lambda_1\dot{\epsilon})(1 + \lambda_1\epsilon)}$$

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- becomes negative just above $\dot{\epsilon} = 1/2\lambda_1$!!!!!

Variants of Oldroyd-B

- White-Metzner to incorporate shear-thinning $\mu(\dot{\gamma})$

$$\sigma + \frac{\mu(\dot{\gamma})}{G} \overset{\triangledown}{\sigma} = 2\mu(\dot{\gamma}) E$$

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- Multi-mode versions of above

Molecular reformulation of Oldroyd-B

also for better numerics

Microstructure A:

$$\overset{\triangledown}{A} + \frac{f}{\tau}(A - I) = 0$$

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FENE modification, for nice behaviour in extensional flow

$$f = \frac{L^2}{L^2 - \mathrm{trace}\ A}$$

K-BKZ model fluid

Kay-Bernstein-Kearsley-Zappa

History dependence through time integrals

K-BKZ model fluid

Kay-Bernstein-Kearsley-Zappa

History dependence through time integrals

Merging of linear viscoelasticity and nonlinear elasticity

$$\sigma = \int_0^\infty \dot{G}(s) \left[\frac{\partial w}{\partial \alpha} \left(\tilde{A}\tilde{A}^T - I \right) - \frac{\partial w}{\partial \beta} \left(\tilde{A}^{-1\,T} \tilde{A}^{-1} - I \right) \right] ds$$

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functions of combinations α and β eigenvalues of $\tilde{A}$.

K-BKZ model fluid 2

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In extensional flow

$$\mu_{\mathrm{ext}} = \int_0^\infty \dot{G}(s)\left[\phi_1\left(e^{2\dot{\epsilon}s} - e^{-\dot{\epsilon}s}\right) + \phi_2\left(e^{\dot{\epsilon}s} - e^{-2\dot{\epsilon}s}\right)\right] s\,ds/\dot{\epsilon}$$

K-BKZ model fluid 3

Wagner model

$$\phi_2 = 0 \quad \text{so } N_2 = 0$$

and

$$\phi_1 = \exp\left(-k\sqrt{\alpha - 3 + \theta(\beta - \alpha)}\right)$$

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In shear shear

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