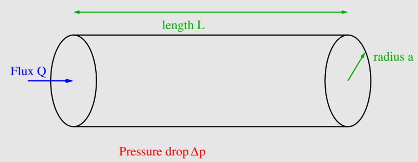
Chapter 4

Some simple flow calculations

- Pipe flow for a power-law fluid
- Capillary rheometry
- Bingham yield fluid in a Couette device
- Rod-climbing
- Unchanging flow field for a second-order fluid
- Converging flow of rigid-rod suspension
- Spinning an Oldroyd-B fluid

Pipe flow for a power-law fluid



Axial momentum

$$0 = -\frac{dp}{dz} + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{zr})$$

so

$$\sigma_{zr} = \frac{r}{R}\sigma_{\text{wall}} \qquad \text{with} \quad \sigma_{\text{wall}} = \frac{\Delta p R}{2L}.$$

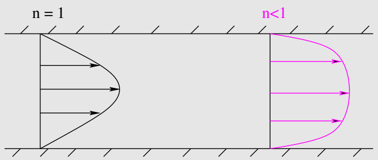
Power-law fluid

$$\sigma_{zr} = \kappa\dot{\gamma}^n \qquad \text{with} \quad \dot{\gamma} = -\frac{dw}{dr}$$

Pipe flow for a power-law fluid 2

Integrating

$$w = \left(\frac{\sigma_w}{\kappa R}\right)^{\frac{1}{n}} \frac{R^{\frac{1}{n}+1} - r^{\frac{1}{n}+1}}{\frac{1}{n}+1}$$



$\begin{cases} \text{Near center, low } \sigma \text{, so high } \mu \\ \text{Near wall, high } \sigma \text{ so low } \mu \end{cases}$ So flattened profile

Hence volume flux

$$Q = \frac{\pi R^3}{\frac{1}{n}+3}\left(\frac{\Delta p R}{2L\kappa}\right)^{\frac{1}{n}}$$

Also wire coating, film draining, drop spreading & peristaltic pumping

Lubrication application: sphere approaching a wall

Gap Sphere radius a, minimum gap d

$$h(r) = d\left(1 + \frac{r^2}{2ad}\right)$$

Mass flux Sphere approaching at velocity W

$$2\pi r Q = \pi r^2 W$$

Power-law flow

$$\frac{dp}{dr} = \frac{\kappa\left(\frac{1}{2}+\frac{1}{4n}\right)^n W^n r^n}{\left[\frac{1}{2}d\left(1+\frac{r^2}{2ad}\right)\right]^{1+2n}}$$

Lubrication application: sphere approaching a wall 2

Force

$$Mg = \kappa \left(\frac{W}{d}\right)^n ad \left(\frac{a}{d}\right)^{\frac{n+1}{2}} \pi 2^{\frac{3n+5}{2}} \left(1+\frac{1}{2n}\right)^n \int_0^\infty \frac{r^{2+n}}{(1+r^2)^{1+2n}}$$

Note integrand like r^{-3n} at large r,
so need $n > \frac{1}{3}$ for lubrication in gap to dominate.

Student Exercise
Find velocity of a sphere falling in a tight tube filled with power-law fluid. *Hint:* $\Delta p \pi a^2 = \Delta\rho \frac{4\pi a^3}{3} g$

Capillary rheometry

Problem: To find $\mu(\dot\gamma)$ even though $\dot\gamma(r)$.

$$\begin{aligned} Q &= \int_0^R w\, 2\pi r\, dr \\ &= -\int_0^R \dot\gamma\, \pi r^2\, dr \qquad \text{as } \frac{dw}{dr} = \dot\gamma \\ &= -\frac{\pi R^3}{\sigma_w^3} \int_0^{\sigma_w} \dot\gamma(\sigma)\sigma^2\, d\sigma \qquad \text{as } \sigma \propto r \end{aligned}$$

Hence

$$\begin{aligned} \dot\gamma_{\text{wall}} &= -\frac{1}{\sigma_w^2} \frac{d}{d\sigma_w} \left(\frac{\sigma_w^3 Q}{\pi R^3}\right) \\ &= -\frac{1}{\pi R^3} \left(3Q + \sigma_w \frac{dQ}{d\sigma_w}\right) \end{aligned}$$

Capillary rheometry 2

So as $\sigma_w \propto \Delta p$

$$\dot\gamma_{\text{wall}} = -\frac{Q}{\pi R^3}\left(3 + \frac{d \ln Q}{d \ln \Delta P}\right),$$

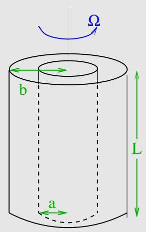
Slope of plot $\ln Q$ vs $\ln \Delta p$, $= 1$ if Newtonian, $= 3$ power-law $n = \frac{1}{3}$.

Then the shear-rate dependent viscosity is found from

$$\mu_w = \frac{\sigma_w}{\dot\gamma_w} = \frac{\Delta p R}{2L\dot\gamma_w}$$

Student Exercise: Similar analysis for a parallel plate rheometer.

Bingham yield fluid in a Couette device



θ-momentum

$$0 = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \sigma_{r\theta}\right) \qquad \text{so} \quad \sigma_{r\theta} = \frac{T}{2\pi L r^2}$$

Bingham fluid

$$\begin{aligned} \dot\gamma &= 0 && \text{if } \sigma < \sigma_Y \\ \sigma_{r\theta} &= \sigma_Y + \mu\dot\gamma && \text{if } \sigma > \sigma_Y \end{aligned}$$

Bingham yield fluid in a Couette device 2

Yields inside surface at

$$r = r_Y = \sqrt{\frac{T}{2\pi L \sigma_Y}}$$

1. All yield $r_Y > b$
2. None yield $r_Y < a$
3. Partial yield $a < r_Y < b$

In $a < r < r_Y$ (yielding)

$$\dot{\gamma} = r\frac{d}{dr}\left(\frac{u_\theta}{r}\right) = \frac{\sigma_Y}{\mu}\left(\frac{r_Y{}^2}{r^2} - 1\right)$$

So

$$\frac{u_\theta}{r} = \frac{\sigma_Y}{\mu}\left[\frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{r^2}\right) - \ln\frac{r}{a}\right]$$

Bingham yield fluid in a Couette device 3

In $r_Y < r < b$ (not yielding)

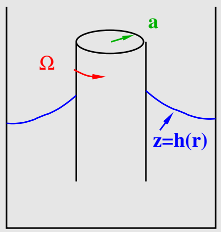
$$\frac{u_\theta}{r} = \Omega$$

Continuity of u_θ at $r = r_Y$ gives

$$\Omega(r_Y(T))$$

Student exercise
Similarly in pipe flow
Similar in squeeze film, although too difficult for a few lectures.

Rod-climbing for a second-order fluid



Flow $\approx$ Newtonian

$$u_\theta = \frac{\Omega a^2}{r} \qquad \text{so} \quad \dot{\gamma} = r\frac{d}{dr}\left(\frac{u_\theta}{r}\right) = -\frac{2\Omega a^2}{r^2}$$

Rod-climbing for a second-order fluid 2

Second-order fluid

$$\sigma = -pI + 2\mu E - 2\alpha \overset{\triangledown}{E} + \beta E \cdot E$$

So

$$\begin{aligned}
\sigma_{\theta r} &= \mu\dot{\gamma} \\
\sigma_{rr} &= -p + \tfrac{1}{4}\beta\dot{\gamma}^2 \\
\sigma_{\theta\theta} &= -p + \left(2\alpha + \tfrac{1}{4}\beta\right)\dot{\gamma}^2 \\
\sigma_{zz} &= -p
\end{aligned}$$

To find $p(r)$ and hence $h(r)$

Rod-climbing for a second-order fluid 3

Radial momentum

$$0 = \frac{\partial \sigma_{rr}}{\partial r} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}, \qquad \text{last term} \quad = -\frac{2\alpha\dot{\gamma}^2}{r} = -\frac{8\alpha\Omega^2 a^4}{r^5}$$

So

$$\left(\sigma_{rr} = -p + \tfrac{1}{4}\beta\dot{\gamma}^2\right) + \tfrac{1}{2}\alpha\dot{\gamma}^2 = f(z)$$

Vertical momentum

$$0 = \frac{\partial \sigma_{zz}}{\partial z} - \rho g, \qquad \text{with} \quad \sigma_{zz} = 0 \text{ on } z = h(r)$$

Hence

$$p = -\sigma_{zz} = \rho g\,(h(r) - z)$$
$$\text{so } h(r) = \frac{1}{\rho g}\,(2\alpha + \beta)\,\frac{\Omega^2 a^4}{r^4}.$$

Could add surface tension and inertia

Unchanging flow field for a second-order fluid

Second-order fluid = Newtonian with small non-linear correction.
Student exercise Show

$$\nabla \cdot \left(2\overset{\triangledown}{E} + 4E \cdot E\right) = \frac{D}{Dt}\nabla^2\mathbf{u} + \nabla\mathbf{u} \cdot \nabla^2\mathbf{u} + \nabla(E : E)$$

If $\mathbf{u}(\mathbf{x},t)$ and $p_1(\mathbf{x},t)$ satisfy Newtonian Stokes flow

$$0 = -\nabla p_1 + \mu\nabla^2\mathbf{u} \qquad \text{and } \nabla \cdot \mathbf{u} = 0,$$

then same $\mathbf{u}(\mathbf{x},t)$ with different $p_2(\mathbf{x},t)$ satisfies (Giesekus) second-order fluid equation

$$\nabla \cdot \sigma = 0$$
$$\sigma = -p_2 + 2\mu E - 2\alpha\overset{\triangledown}{E} + \beta E \cdot E$$

with

$$\beta = -4\alpha \quad \text{and } p_2 = p_1 - \frac{\alpha}{\mu}\frac{Dp_1}{Dt} + \alpha E \cdot E \qquad \text{Student Exercise}$$

Unchanging flow field for a second-order fluid 2

Similar results with no restriction of α and β

- Planar flows – Tanner & Pipkin
- unidirectional flows – Langlois, Rivlin & Pipkin

Converging flow of rigid-rod suspension

Rheology: an anisotropic viscosity in direction of rods/fibres $\mathbf{p}$

$$\sigma = -pI + 2\mu_{\text{shear}}E + 2\mu_{\text{ext}}\mathbf{pp}(\mathbf{p} \cdot E \cdot \mathbf{p})$$

In 2-D sink flow, radial flow $u_r = f(\theta)/r$ and rods align radially $p_r = 1$.

So with pressure $g(\theta)/r^2$ the stress is

$$\sigma_{rr} = -\frac{g}{r^2} - 2(\mu_s + \mu_e)\frac{f}{r^2}, \qquad \sigma_{r\theta} = \mu_s\frac{f'}{r^2}, \qquad \sigma_{\theta\theta} = -\frac{g}{r^2} + 2\mu_s\frac{f}{r^2}.$$

Converging flow of rigid-rod suspension 2

θ-momentum

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0$$

so

$$g' = \mu_s f'$$

Radial momentum

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0$$

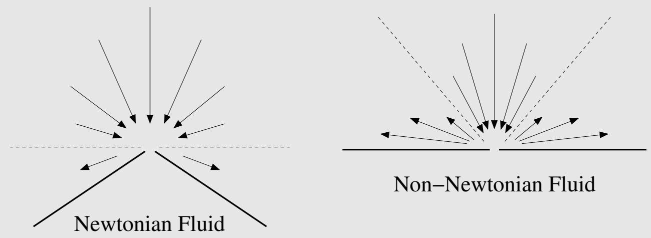
so

$$f'' + \left(4 + 2\frac{\mu_e}{\mu_s}\right) f = \mathrm{const}$$

A compression in θ-direction of $\sqrt{1 + \mu_e/2\mu_s}$

Converging flow of rigid-rod suspension 3

Newtonian flow has recirculation region if angle $> \pi$

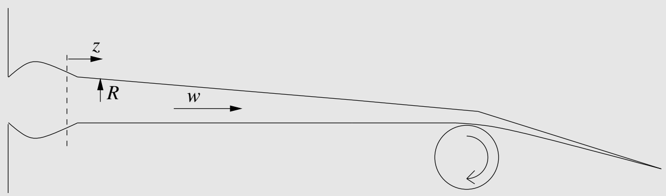


Rigid-rod suspension, with the compression in θ-direction, has recirculation region at angle $= \pi$

Anisotropy in rheology leads to anisotropy in flow

Also 3D sink flow.
Also flow round a sharp corner (rods along streamlines).

Spinning an Oldroyd-B fluid



Volume flux

$$Q = \pi R^2 w$$

Tension, ignoring surface tension, gravity and inertia

$$F = \pi R^2 \sigma_{zz}$$

Oldroyd-B

$$\sigma = -pI + 2\mu E + GA$$

$$\frac{DA}{Dt} = A \cdot \nabla u + \nabla u^T \cdot A - \frac{1}{\tau}(A - I)$$

Spinning an Oldroyd-B fluid 2

So

$$w\frac{dA_{rr}}{dz} = -A_{rr}\frac{dw}{dz} - \frac{1}{\tau}(A_{rr} - 1)$$
$$w\frac{dA_{zz}}{dz} = 2A_{zz}\frac{dw}{dz} - \frac{1}{\tau}(A_{zz} - 1)$$

Free surface

$$\sigma_{rr} = 0, \qquad \text{so} \quad p = -\mu\frac{dw}{dz} + GA_{rr}$$

Momentum equation

$$\sigma_{zz} = 3\mu\frac{dw}{dz} + G(A_{zz} - A_{rr}) = \frac{F}{\pi R^2} = \frac{Fw}{Q}$$

This equation gives dw/dz which the can use in $dA_{..}/dz$ equations above.

Spinning an Oldroyd-B fluid 3

Newtonian limit $\tau dw/dz \ll 1$

$$A_{rr} \sim 1 - \tau \frac{dw}{dw}, \quad A_{zz} \sim 1 + 2\tau \frac{dw}{dz}$$

so

$$\sigma_{zz} \sim 3(\mu + G\tau)\frac{dw}{dz} = \frac{Fw}{Q}$$

so

$$w(z) \sim w(0) \exp\left(\frac{Fz}{3Q(\mu + G\tau)}\right)$$

Spinning an Oldroyd-B fluid 4

Elastic limit $\mu dw/dz, GA_{rr} \ll GA_{zz}$

$$\frac{Fw}{Q} = \sigma_{zz} \sim GA_{zz}, \quad \text{or } A_{zz} \sim \frac{Fw}{GQ}$$

substitute into

$$w\frac{dA_{zz}}{dz} = 2A_{zz}\frac{dw}{dz} - \frac{1}{\tau}(A_{zz} - 1(\leftarrow \text{small}))$$

for

$$w\frac{dw}{dz} = 2w\frac{dw}{dz} - \frac{1}{\tau}w$$

with solution

$$w = w_0 + \frac{z}{\tau}, \qquad \text{independent of } F\ !$$

Need stretch to avoid relaxation