No lecture Thursday 17 February 2011

Next lecture Tuesday 22 February

Discretisation

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  - good for complex geometry
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  - commercial code : POLYFLOW

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- Finite differences
  - simple, so good for understanding underlying difficulties
  - only for simple geometry (but mappable)

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E.G. for a triangle $(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$,
$\phi_1(\mathbf{x}) = 1$ at vertex $\mathbf{x} = \mathbf{x}_1$ and vanishing at $\mathbf{x}_2$ and $\mathbf{x}_3$

$$\phi_1(\mathbf{x}) = \frac{(\mathbf{x} - \mathbf{x}_2) \times (\mathbf{x}_3 - \mathbf{x}_2) \cdot \hat{\mathbf{z}}}{(\mathbf{x}_1 - \mathbf{x}_2) \times (\mathbf{x}_3 - \mathbf{x}_2) \cdot \hat{\mathbf{z}}}$$

Finite Elements 2

- Substitute into momentum/mass/stress equation and project (Galerkin)

$$\int \left(\rho \frac{Du}{Dt} - \nabla \cdot \sigma \right) \cdot \phi_s(\mathbf{x})\, dV \;=\; 0, \qquad s \;=\; 1, 2, .., N$$

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- Typical finite elements have less pressure modes than velocity, and sometimes more stress than velocity

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- So use pseudo-spectral – evaluate products in real space and derivatives in Fourier space.

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- Aliasing – chop top $\frac{1}{3}$ of spectrum

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  - gives organised labelling
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- Conservative, e.g.

$$\nabla^4\psi = \nabla \times \nabla \cdot (\nabla + \nabla^T)\nabla \times \psi \neq \nabla^2\nabla^2\psi$$

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Pressure ensures incompressibility

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Half step to u^* using no-slip BC

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Also pressure update $O(\Delta t^2)$

FD pressure problems

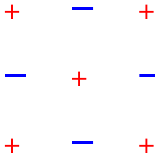
Spurious pressure modes

+	−	+
−	+	−
+	−	+

“$\nabla p = 0$”

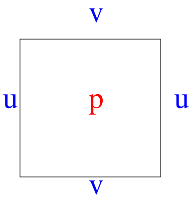
FD pressure problems

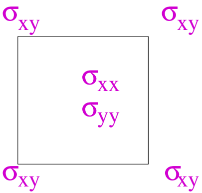
Spurious pressure modes



"$\nabla p = 0$"

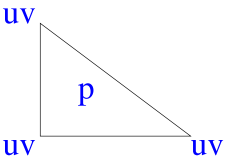
Avoided by staggered grid





FE pressure problems

- Spurious pressure modes with "$\nabla p = 0$" – no staggered FE



One Δ has $1p + 3u + 3v$

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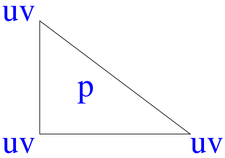
- Locking



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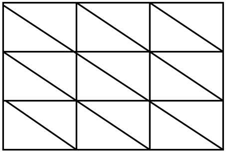

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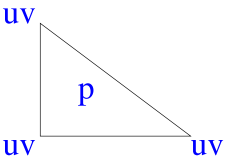

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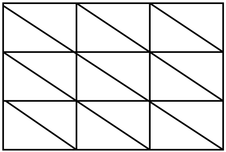

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Use 'bubble elements' with extra u, v at centre of triangles

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Write EVSS = Elastic Viscous Split Stress

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- domain decomposition

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- Fast relaxed modes

$$\mu = \mu_0 + \sum_{\tau_i \ll \dot{\gamma}^{-1}} G_i \tau_i$$

Hyperbolic part

Stress equation is hyperbolic PDE

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- second-order with 'flux-limiters', e.g. MINMOD
- use characteristics = streamlines

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Finite Elements

- PUPG – Streamline Upwinding Petrov Galerkin:

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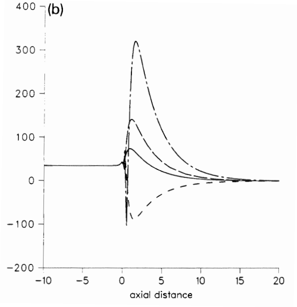
  - no fast elliptic solver

Hyperbolic part 3

Typical erroneous treatment of hyperbolic stress equation



Continuous curve is correct solution.
Others have spurious oscillations.

Bench marks

International campaign tackling bench-mark problems

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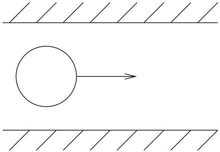
1. Sphere in a tube, 2:1 diam
Dominated by shear



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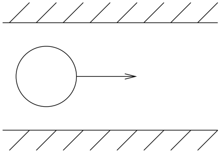
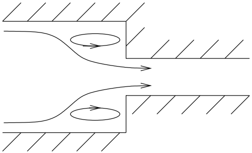


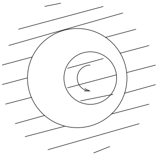
2. Contraction, 4:1
Difficult sharp corner



Bench marks 2

3. Journal bearing

Good for spectral



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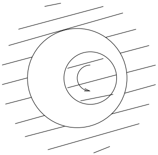

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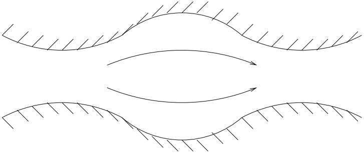

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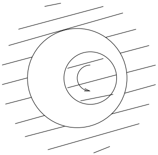

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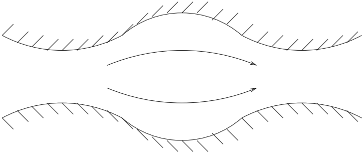

Eventually different algorithms produced the same results!

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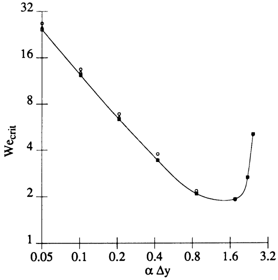
- Limiting (maximum) value of *De*, e.g. sphere in a tube:
  - UCM $De_{\mathrm{max}} = 2.17$
  - O-B $De_{\mathrm{max}} = 1.28$ Fan (2003) JNNFM 110

Numerical problems 2

New numerical instability

Plotting σ_{xx}/σ_{xy} vs $\Delta y/\Delta x$

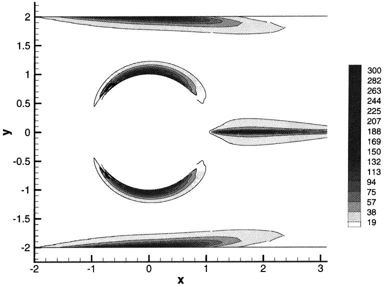


Need $\quad \Delta y < \Delta x \dfrac{\sigma_{xy}}{\sigma_{xx}} \quad$ to resolve direction of large N_1

Numerical problems 3

Thin layers of high stress

Flow past a sphere in a tube



Need to resolve

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- Micro-Macro Brownian fields, with same random Brownian forces in all spatial blocks, see later