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- Either ‘ad hoc’, such as Oldroyd-B differential equation and BKZ integral equation,
- Or look at microstructure for highly idealised systems and derive their constitutive equations.
- Most will be suspensions of small particles in Newtonian viscous solvent.

Microstructural studies for rheology

- Micro & macro views
- Einstein viscosity
- Rotations
- Deformations
- Interactions
- Polymers
- Others

Micro & macro views

- Separation of length scales
- Micro $\leftrightarrow$ Macro connections
- Case of Newtonian solvent
- Homogenisation

Separation of length scales

Essential

$$\text{Micro} \quad \ell \ll L \quad \text{Macro}$$

$$\text{Micro} = \text{particle } 1\mu m \qquad \text{Macro} = \text{flow, } 1cm$$

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  But macro-Reynolds number $Re_L = \frac{\rho\gamma L^2}{\mu}$ can be large
- If $\ell \not< L$, then non-local rheology

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Here: suspension of particles in Newtonian viscous solvent

1. Macro→micro connection

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both needing $Re_\ell \ll 1$.

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To be used in averaged = macro momentum equation

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NB micro-Reynolds stresses $\overline{(\rho \mathbf{u})' \mathbf{u}'}$ small for $Re_\ell \ll 1$.

Reduction for suspension with Newtonian viscous solvent

Write:

$$\sigma = -pI + 2\mu e + \sigma^+$$

with pressure p, solvent viscosity μ, strain-rate e, and σ^+ non-zero only inside particles.

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$$\overline{\sigma^+} = \frac{1}{V}\int_V \sigma^+ \, dV = n \left\langle \int_{\text{particle}} \sigma^+ \, dV \right\rangle_{\text{types of particle}}$$

with n number of particles per unit volume

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Integral called 'stresslet', is the force-dipole strength of the particle.

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Multiscale asymptotic expansion

$$T(x;\epsilon) \sim T_0(x,\xi) + \epsilon T_1(x,\xi) + \epsilon^2 T_2(x,\xi)$$

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ϵ^{-2}:

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Thus T varies only slowly at leading order, with microscale making small perturbations.

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$$T_1(x,\xi) = A(\xi)\partial_x T_0$$

Homogenisation 4

ϵ^0:

$$\partial_\xi k \partial_\xi T_2 = Q - \partial_x k \partial_x T_0 - \partial_\xi k \partial_x T_1 - \partial_x k \partial_\xi T_1$$

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Hence macro description

$$\nabla k^* \nabla T = Q^* \qquad \text{with} \quad k^* = \left\langle k + k \frac{\partial A}{\partial \xi} \right\rangle \quad \text{and} \quad Q^* = \langle Q \rangle$$

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Hence heat flux

$$\langle q \rangle = \langle k \nabla T_{\text{micro}} \rangle = \langle k + \epsilon k \nabla A \rangle \nabla T_0$$

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Microstructural studies for rheology

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- Einstein viscosity
- Rotations
- Deformations
- Interactions
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Stokes problem for Einstein viscosity

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$$\mathbf{u} = \mathbf{V} + \omega \times \mathbf{x} \quad \text{on} \quad r = a \quad \text{with} \quad V, \omega \text{ consts}$$
$$\mathbf{u} \to \overline{U} + \mathbf{x} \cdot \nabla \overline{U} \quad \text{as} \quad r \to \infty$$

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NB: Stokes problem is linear and instantaneous   Student Ex

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Then S.Ex

$$\mathbf{u} = \overline{U} + \mathbf{E} \cdot \mathbf{x} + \Omega \times \mathbf{x} - \mathbf{E} \cdot \mathbf{x} \frac{a^5}{r^5} - \mathbf{x} \frac{5(\mathbf{x} \cdot \mathbf{E} \cdot \mathbf{x})}{2r^2} \left(\frac{a^3}{r^3} - \frac{a^5}{r^5} \right)$$

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Evaluate particle contribution to macro/average stress

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Result for Einstein viscosity (1905)

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  which arbitrarily cancelled divergent integrals (hydrodynamics is long-ranged)

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