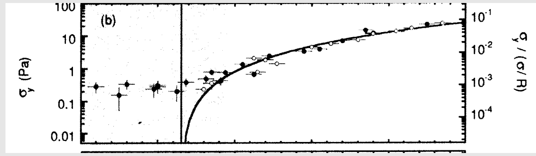
Chapter 8

Yield problems

- Yield stress
  - foams
  - cross-linked gels
  - pastes
- Simple applications
  - transport of small particles
  - dangerous no-flow in quiet corners
- Squeeze film paradox
- Ketchup bottle & oil pipelines

Yield Stress in foams

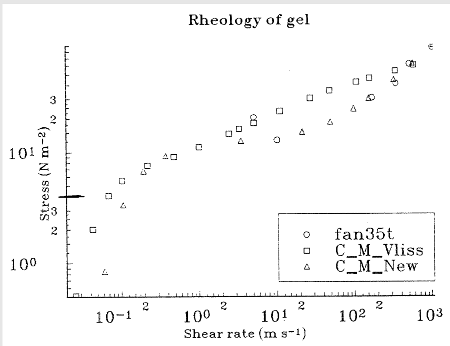
Yield stress vs volume fraction 0.5 to 1.0, curve $0.73(\phi - \phi_c)^2$.



St James & Durian (1999) J.Rheol 43

Foams permanently damages where yield? Debregeas (2003)

Yield stress of a gel



Yield stress of a paste

Stress as function of shear-rate at different pH.
Suspension of $0.33\mu m$ aluminium particles at $\phi = 0.3$

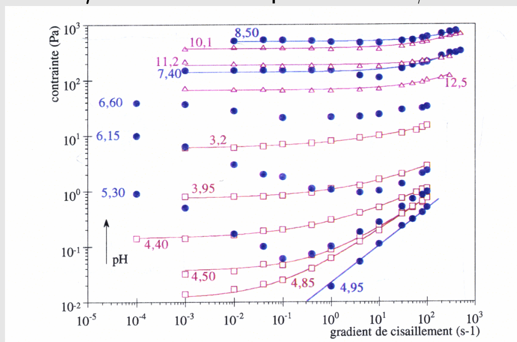


Fig.3 : Courbes d'écoulement de suspensions d'alumine P772SB, en fonction du pH, ϕ_v=0,30.

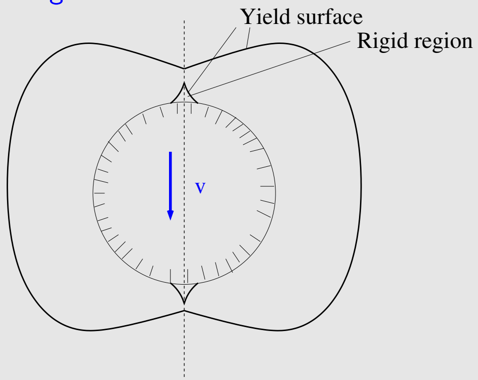
Ducerf (Grenoble PhD 1992)

Note yield stress very sensitive to pH

Sedimentation (or NOT)

Need $\frac{F}{4\pi a^2} > 3.5\sigma_Y$

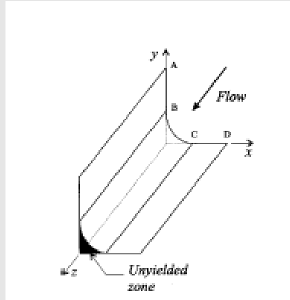
Flow in a finite region



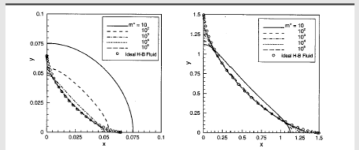
Beris, Tsamopoulos & Armstrong (1985) JFM 158

Use to transport particles without sedimentation

No flow in quiet corners



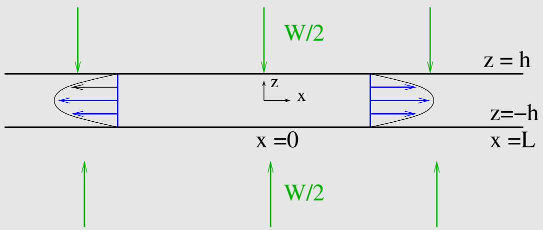
Burgos, Alexandrou & Entov (1999) J.Rheol 43



Bingham yield fluid, $\sigma_Y H/\mu U = 0.1,\ 10$

Squeeze film paradox

Walton & Bittleston (1991) JFM 222 – pseudo & real plugs
Balmforth & Craster (1999) JNNFM 84 – lubrication thy correct
Wilson (1993) JNNFM 47 – 2 limits in bi-viscosity



Squeeze film 2 – the problem

Momentum

$$0 = -p_x + \sigma_{xx,x} + \sigma_{xz,z}$$
$$0 = -p_z + \sigma_{xz,x} + \sigma_{zz,z}$$

Rheology – Bingham

$$\begin{cases} E = 0 & \text{if } |\sigma| < \sigma_Y \\ \sigma = \left(2\mu + \frac{\sigma_Y}{|E|}\right) E & \text{if } |\sigma| > \sigma_Y \end{cases}$$

where $|E| = \sqrt{\frac{1}{2}E : E}$, $|\sigma| = \sqrt{\frac{1}{2}\sigma : \sigma}$,

$$E = \begin{pmatrix} u_x & \frac{1}{2}(u_z + w_x) \\ \frac{1}{2}(u_z + w_x) & w_z \end{pmatrix}$$

Squeeze film 3 – nondimensionalised

As for Newtonian lubrication:
x on L, z on H, w on W, u on WL/H, E on W/H
σ_{xz}, σ_Y on $\mu WL/H^2$, σ_{xx}, σ_{zz} on $\mu W/H$, p on $\mu WL^2/H^3$

Then with $\epsilon = H/L$

$$0 = -p_x + \epsilon^2 \sigma_{xx,x} + \sigma_{xz,z}$$
$$0 = -\epsilon^{-2} p_z + \sigma_{xz,x} + \sigma_{zz,z}$$

so

$$p = p(x) + O(\epsilon^2)$$

$$E = \begin{pmatrix} \epsilon u_x & \frac{1}{2}(u_z + \epsilon^2 w_x) \\ \frac{1}{2}(u_z + \epsilon^2 w_x) & \epsilon w_z \end{pmatrix}$$

So leading order rheology

$$\begin{cases} u_z = 0 & \text{if } |\sigma_{xz}| < \sigma_Y \\ \sigma_{xz} = \mp\sigma_Y + \pm u_z \quad \text{in } z \gtrless 0 & \text{if } |\sigma_{xz}| > \sigma_Y \end{cases}$$

Squeeze film 4 – profile

Integrate x-momentum

$$\sigma_{xz} = \frac{dp}{dx} z + O(\epsilon^2)$$

Hence yields level Y

$$z = Y = \sigma_Y / \left(-\frac{dp}{dx}\right) \qquad \text{(Yld)}$$

Rheology

$$u_z = \begin{cases} 0 & \text{in } 0 \le z \le Y \\ \frac{dp}{dx}(z - Y) & \text{in } Y \le z \le 1 \end{cases}$$

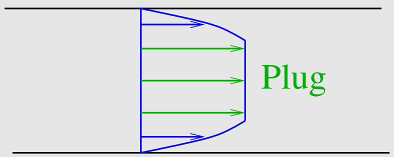
Hence velocity profile

$$u = \begin{cases} U & \text{in } 0 \le z \le Y \\ U + \frac{dp}{dx}\frac{1}{2}(z - Y)^2 & \text{in } Y \le z \le 1 \end{cases}$$

Then no-slip $u = 0$ and $z = 1$ for dp/dx

Squeeze film 5 – profile 2

$$u = \begin{cases} U & \text{in } 0 \le z \le Y \\ U\left(1 - \frac{(z - Y)^2}{(1 - Y)^2}\right) & \text{in } Y \le z \le 1 \end{cases}$$



Flux

$$Q = \int_0^1 u\,dz = U\left(\tfrac{2}{3} + \tfrac{1}{3}Y\right)$$

Mass conservation

$$Q = \tfrac{1}{2}x$$

Hence... U and then dp/dx

Squeeze film 6 – solved

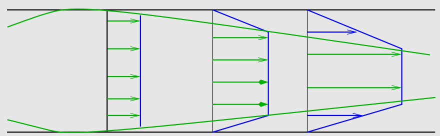
$$U = \tfrac{1}{2}x / \left(\tfrac{2}{3} + \tfrac{1}{3}Y\right)$$

$$\text{with } \frac{dp}{dx} = -x / \left(\tfrac{2}{3} + \tfrac{1}{3}Y\right)(1 - Y)^2$$

Substituting this pressure gradient into the yield condition (Yld):

$$Y = \left(\tfrac{2}{3} + \tfrac{1}{3}Y\right)(1 - Y)^2 / x \quad \sim \quad \begin{cases} 1 - x^{1/2} & x \ll 1 \\ 2/3x & x \gg 1 \end{cases}$$

Paradox: plug velocity varies $U(x)$



Squeeze film 7 – paradox resolved

Can have $U(x)$ in a pseudo-plug if just above yield:

$$|\sigma| = \sigma_Y + O(\epsilon).$$

Now

$$\sigma_{xz} = \sigma_Y z/Y \quad \text{in } 0 \leq z \leq Y,$$

so need

$$\sigma_{xx} = -\sigma_{zz} = \sigma_Y \sqrt{1-(z/Y)^2}$$

What flow does this stress drive in the plug?

Squeeze film 8 – paradox resolved

Straining $E \propto \sigma$, so $u = U(x) + \epsilon u_1(x,z)$

$$E \sim \epsilon \begin{pmatrix} U_x & \frac{1}{2}u_{1z} \\ \frac{1}{2}u_{1z} & U_x \end{pmatrix}, \quad \text{so } |E| = \epsilon\sqrt{U_x^2 + \tfrac{1}{4}u_{1z}^2}.$$

Then

$$-\sigma_Y z/Y = \sigma_{xz} = \left(2 + \frac{\sigma_Y}{|E|}\right)\epsilon\tfrac{1}{2}u_{1z}$$

i.e.

$$u_{1z} = -\frac{2U_x z/Y}{\sqrt{1-(z/Y)^2}} \quad \text{(singular as } z \nearrow Y\text{)}$$

so

$$u_1 = 2U_x Y\sqrt{1-(z/Y)^2}$$

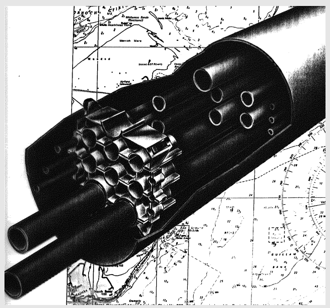
(Singularity gives $O(\epsilon)$ transition layer at $z = Y$)

Bottom line: naive plug works, even if it is a pseudo-plug

Ketchup bottle problem

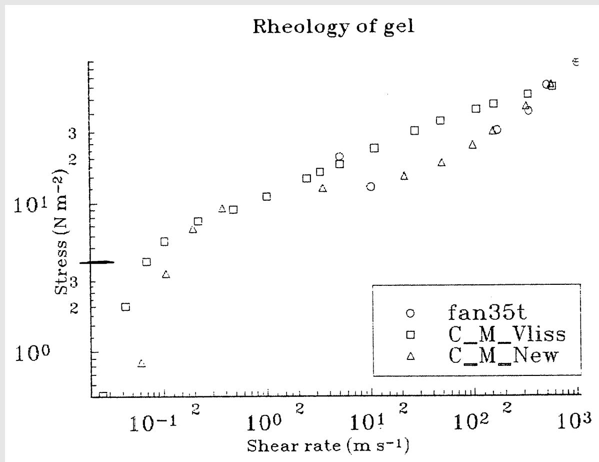
with applications to oil pipeline assembles

North Sea. Shell Gannet project



Costain

Ketchup bottle – rheology of gel



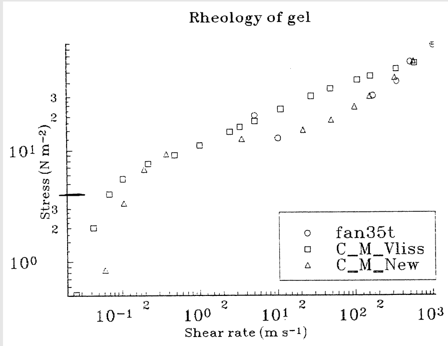
Ketchup bottle – three questions

- Will the gel convect at $\Delta t = 80^{\circ}\mathrm{C}$?
- Pressure to pump 3km in 10 hours?
- How much flows out of the ketchup bottle?

Ketchup bottle – the gel

Observe 1mm bubbles do not move.
Hence yield stress is

$$\sigma_Y = \rho g d = 10\,\mathrm{Pa}.$$



Ketchup bottle – convection?

$$\Delta T = 80^{\circ}\mathrm{C} \quad \rightarrow \quad \frac{\Delta\rho}{\rho} = 10^{-2}$$

Hence 10cm "bubble" will not move by yield stress.

Answer1: will not convect

Ketchup bottle – pumping

Pipe radius r, length L.
Pressure drop balancing yield stress

$$\pi r^2 \Delta p = 2\pi r L \sigma_Y$$

Hence

$$\frac{\Delta p}{L} = 200\,\mathrm{Pa\,m^{-1}}$$

(measure 350 in 100m test)

Hence

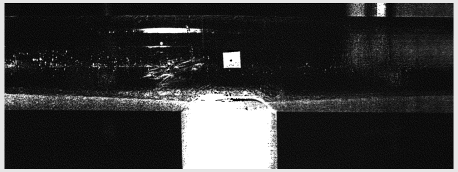
$$\Delta p = 6 \text{ bar in 3km}$$

Strength of pipe 50 bar.

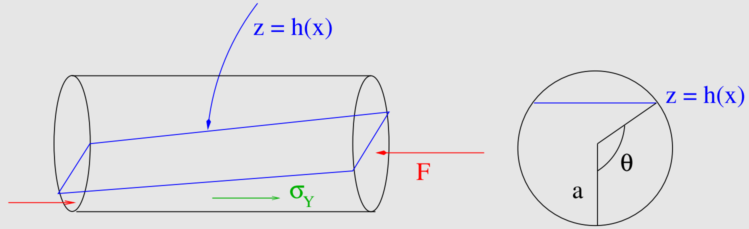
Answer2: safe to pump

Ketchup bottle problem

- Hot oil in pipe
  - gel expands
    - gel flows into special expansion pipe
- Production stops
  - gel cools and contracts
    - Does gel come out of expansion pipe?



Ketchup bottle – idealised bottle



Pressure force

$$F = \int p \, dA \quad \text{with } p = \rho g (h - z)$$

Gradient balances the wall stress, all at yield

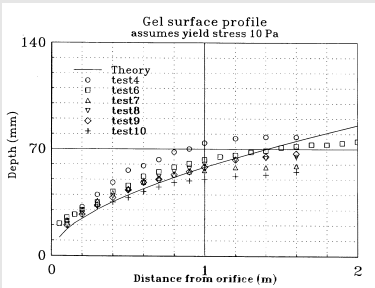
$$\sigma_Y 2a\theta = \frac{dF}{dx} = \rho g \frac{dh}{dx} A$$

Now $h = a(1 - \cos\theta)$ and $A = a^2(\theta - \frac{1}{2}\sin 2\theta)$,

Ketchup bottle – idealised continued

so

$$\frac{d\theta}{dx} = \frac{\sigma_Y}{\rho g a^2} \frac{2\theta}{\sin\theta(\theta - \frac{1}{2}\sin 2\theta)}$$



Hence volume removed $1.69a^3 \frac{\rho g a}{\sigma_Y}$ from length $0.85a \frac{\rho g a}{\sigma_Y}$

Answer3: enough flows out of the bottle, just

Student Exercise

Consider a gravity current of a yield fluid on an inclined plane.