Mathematical Tripos Part II(B) E.J. Hinch
Fluid Dynamics (B20) November 1998

Example Sheet 3

1. The walls of a channel are porous and separated by a distance d. Fluid is driven through the channel by a pressure gradient $G = -\partial p/\partial x$, and at the same time suction is applied to one wall of the channel providing a cross flow with uniform transverse component of velocity V, fluid being supplied at this rate at the other wall. Find and sketch the steady velocity and vorticity distributions in the fluid (i) when $Vd/\nu \ll 1$ and (ii) when $Vd/\nu \gg 1$.

2. Viscous fluid fills an annulus $a < r < b$ between a long stationary cylinder $r = b$ and a long cylinder $r = a$ rotating at angular velocity Ω. Find the axisymmetric velocity field, ignoring end effects.

Suppose now that the two cylinders are porous, and a pressure difference is applied so that there is a radial flow $-Va/r$. Find the new steady flow around the cylinder when $Va/\nu < 2$ and $Va/\nu > 2$. Comment on the flow structure when $Va/\nu \gg 1$.

Find the torque that must be applied to maintain the motion.

3. Starting from the Navier-Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

Interpret the terms in the equation.

At time $t = 0$ a concentration of vorticity is created along the z-axis, with the same circulation Γ around the axis at each z. The fluid is viscous and incompressible, and for $t > 0$ has only an azimuthal velocity v, say. Show that there is a similarity solution of the form $vr/\Gamma = f(\eta)$, where $r = (x^2 + y^2)^{1/2}$ and η is a suitable similarity variable. Further show that all conditions are satisfied by

$$f(\eta) = \tfrac{1}{2\pi}(1 - e^{-\eta}), \qquad \eta = r^2/4\nu t.$$

Show also that the total vorticity in the flow remains constant at Γ for all $t > 0$. Sketch v as a function of r.

4. Calculate the vorticity $\boldsymbol{\omega}$ associated with the velocity field

$$\mathbf{u} = (-\alpha x - yf(r,t),\, -\alpha y + xf(r,t),\, 2\alpha z)\,,$$

where α is a positive constant, and $f(r,t)$ depends on $r = (x^2 + y^2)^{1/2}$ and time t. Hence show that the velocity field represents a dynamically possible motion if $f(r,t)$ satisfies

$$2f + r\frac{\partial f}{\partial r} = A\gamma(t)e^{-\gamma(t)r^2},$$

where

$$\gamma(t) = \frac{\alpha}{2\nu}\left(1 \pm e^{-2\alpha(t-t_0)}\right)^{-1},$$

and A and t_0 are constants.

Show that in the case where the minus sign is taken γ is approximately $1/[4\nu(t-t_0)]$ when t only just exceeds t_0. Which terms in the vorticity equation dominate when this approximation holds?

5. Wind blowing over a reservoir exerts at the water surface a uniform tangential stress S which is normal to, and away from, a straight side of the reservoir. Use dimensional analysis, based both on balancing the inertial and viscous forces in a thin boundary layer and on the imposed boundary condition, to find order-of-magnitude estimates for the boundary-layer thickness $\delta(x)$ and the surface velocity $U(x)$ as functions of distance x from the shore. Using the boundary-layer equations, find the ordinary differential equation governing the non-dimensional function f defined by

$$\psi(x,y) = u(x)\delta(x)f(\eta), \qquad \text{where} \quad \eta = y/\delta(x).$$

What are the boundary conditions on f?

6. A steady two-dimensional jet of fluid runs along a plane rigid wall, the fluid being at rest far from the wall. Use the boundary-layer equations to show that the quantity

$$P = \int_0^\infty u(y) \left(\int_y^\infty u(y')^2 \, dy' \right) dy$$

is independent of the distance x along the wall. Find order-of-magnitude estimates for the boundary-layer thickness and velocity as functions of x.

Show that in the analogous axisymmetric wall jet spreading out radially the velocity varies like $r^{-3/2}$.

7. Show that the streamfunction $\psi(r,\theta)$ for a steady two-dimensional flow satisfies

$$-\frac{1}{r}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r,\theta)} = \nu\nabla^4\psi.$$

Show further that this equation admits solutions of the form

$$\psi = Qf(\theta),$$

if f satisfies

$$f'''' + 4f'' + \frac{2Q}{\nu} f' f'' = 0.$$

[See lectures for solutions.]

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