Example Sheet 4

1. A plank of width $2a$ is held horizontal and dropped onto the surface of water filling the half-space $y < 0$. Let U be the speed of fall when the plank hits the water. Show that the instantaneous flow of the water can be represented by the complex potential

$$w(z) = iU\left(z - \left(z^2 - a^2\right)^{1/2}\right),$$

and determine the instantaneous velocity of particles in the free surface $y = 0$. Calculate the pressure field in the water at the instant of impact, and show that the force experienced by the plank at this instant is infinite. Calculate too the magnitude of the impulse suffered by the plank. *[The time-integral of the pressure across a discontinuous change of velocity is* $-\rho[\phi]_-^+$*.]*

2. A cylinder, boundary C, is immersed in a flow with complex potential $w(z)$. Show that the force (F_x, F_y) and couple G acting on the cylinder is given by

$$F_x - iF_y \;=\; i\int_C \tfrac{1}{2}\rho\left(\frac{dw}{dz}\right)^2 dz \quad \text{and} \quad G \;=\; -\mathrm{Re}\int_C \tfrac{1}{2}\rho\left(\frac{dw}{dz}\right)^2 z\,dz.$$

[Note $(n_x - in_y)dl = i\overline{dz}$ *and along a streamline* $\overline{dw} = dw$*.]*

Suppose now that there are no singularities of the integrands in the region occupied by the fluid and that as $z \to \infty$

$$w \;\sim\; Uz - \frac{i\kappa}{2\pi}\ln z + \frac{D}{z},$$

where κ is real but U and D are complex. By deforming the contour C to a large circle, evaluate F and G.

3. Use question 2 to find the torque on a flat plate for which the circulation κ satisfies the Kutta condition. Where on the plate must the lift force be deemed to act so as to guarantee the correct torque about the centre of the plate?

4. An elliptical cylinder $\frac{1}{4}x^2 + y^2 = \alpha^2$ lies in a uniform stream of speed $U > 0$ in the x-direction, and there is a circulation $\kappa > 0$ about the cylinder. Use a conformal transformation of the form

$$z \;=\; \zeta \;+\; \frac{m^2\alpha^2}{\zeta},$$

where m is a number to be calculated, to transform the ellipse to a circle, and hence obtain the complex velocity potential of the flow. Show that the stagnation points are given in general by

$$z = i\beta \pm 2\left(\alpha^2 - \beta^2\right)^{1/2},$$

where $\beta = \kappa/6\pi U$, and examine the position of the stagnation points in the cases

(i) $\alpha > \beta$, (ii) $\alpha = \beta$, (iii) $\alpha < \beta < 5\alpha/3$, (iv) $\beta > 5\alpha/3$.

5. A vortex sheet of strength U is located at a distance h above a rigid wall $y = 0$ and is parallel to it, so that the fluid velocity $(u, 0, 0)$ is

$$u \;=\; \begin{cases} U & \text{in } 0 < y < h, \\ 0 & \text{in } y > 0\ . \end{cases}$$

Suppose now that the sheet is perturbed slightly to the position $y = h + \eta_0 e^{ik(x-ct)}$ where $k > 0$ is real but c may be complex. Show that

$$c = U/(1 \pm i\sqrt{\tanh kh}).$$

Deduce that
(i) the sheet is unstable to disturbances of all wavelengths;
(ii) for short waves ($kh \gg 1$) the growth rate $k\mathrm{Im}(c)$ is $\frac{1}{2}Uk$ and the wave propagation speed $\mathrm{Re}(c)$ is $\frac{1}{2}U$, as if the wall were absent;
(iii) for long waves ($kh \ll 1$) the growth rate is $Uk\sqrt{kh}$ (so that the wall inhibits the growth of long waves) and the propagation speed is U.

6. A two-dimensional jet in the x-direction has velocity profile

$$u \;=\; \begin{cases} 0 & \text{in } y > h, \\ U & \text{in } -h < y < y, \\ 0 & \text{in } y < -h\ . \end{cases}$$

The vortex sheets at $y = \pm h$ are perturbed to

$$y \;=\; \begin{cases} +h + \eta_1 e^{ik(x-ct)}, \\ -h + \eta_2 e^{ik(x-ct)}. \end{cases}$$

Show that the jet is unstable to a 'varicose' instability for which $\eta_1 = -\eta_2$ (identical to that of question 5), and also to a 'sinuous' instability for which $\eta_1 = \eta_2$ and

$$c = U/(1 \pm i\sqrt{\coth kh}).$$

[The growth rates at small kh are again $Uk\sqrt{kh}$. Hence thin jets (e.g. smoke filaments) can suffer rather slowly growing sinuous instabilities.]

7. Show that the rate of dissipation of mechanical energy in an incompressible fluid is $2\mu e_{ij}e_{ij}$ per unit volume, where e_{ij} is the rate-of-strain tensor and μ is the viscosity.

A finite mass of incompressible fluid, of viscosity μ and density ρ is held in the shape of a sphere $r < a$ by surface tension. It is set into a mode of small oscillations in which the velocity filed may be taken to have Cartesian components

$$u = \beta x, \quad v = -\beta y, \quad w = 0.$$

where $\beta \propto \exp(-\epsilon t) \sin \omega t$. Assuming that $\epsilon \ll \omega$, calculate the dissipation rate averaged over a cycle (ignoring the slowly varying factor $\exp(-\epsilon t)$) and hence show that $\epsilon = 5\mu/\rho a^2$. You may assume that the total energy of the oscillation is twice the kinetic energy averaged over a cycle. Why is is permissible to ignore the details of the boundary layer near $r = a$?

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