Laminar Boundary-Layer Theory: A 20th Century Paradox?

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Introduction

20th Century: Prandtl published his seminal paper in 1904.

Boundary-Layer Theory: He proposed that viscous effects would be confined to thin shear layers in the case of the 'motion of fluids with very little viscosity', i.e. when the Reynolds number is large ($Re \gg 1$).

Laminar: although $Re \gg 1$, Prandtl (and others) studied non-turbulent flows.

A Paradox: laminar flows at asymptotically large Reynolds numbers are unstable.

?: ??

Notation & Assumptions

Typical scales

$\mathcal{U}$: velocity

$\mathcal{T}$: time

$\mathcal{L}$: ‘body’ length

δ: ‘shear-layer’ width

$\mathcal{P}$: pressure

Constants

μ: viscosity

ρ: density (incompressible)

Assumptions

Large Reynolds number: $\boldsymbol{Re} = \frac{\rho\mathcal{U}\mathcal{L}}{\mu} \gg 1$

Thin shear layer: $\delta \ll \mathcal{L}$

2D: for simplicity

Classical Boundary-Layer Theory

Continuity

$$\hat{u}_{\hat{x}} + \hat{v}_{\hat{y}} = 0$$

$$\frac{\mathcal{U}}{\mathcal{L}} \sim \frac{\mathcal{V}}{\delta} \qquad \Rightarrow \mathcal{V} \sim \frac{\mathcal{U}\delta}{\mathcal{L}} \ll \mathcal{U}$$

'Stream-wise' momentum

$$\rho\left(\hat{u}_{\hat{t}} + \hat{u}\hat{u}_{\hat{x}} + \hat{v}\hat{u}_{\hat{y}}\right) = -\hat{p}_{\hat{x}} + \mu\left(\hat{u}_{\hat{y}\hat{y}} + \hat{u}_{\hat{x}\hat{x}}\right)$$

$$\frac{\rho\mathcal{U}}{\mathcal{T}} \sim \frac{\rho\mathcal{U}^2}{\mathcal{L}} \sim \frac{\rho\mathcal{V}\mathcal{U}}{\delta} \sim \frac{\mathcal{P}}{\mathcal{L}} \sim \frac{\mu\mathcal{U}}{\delta^2} \gg \frac{\mu\mathcal{U}}{\mathcal{L}^2}$$

$$\Rightarrow \quad \mathcal{P} \sim \rho\mathcal{U}^2, \quad \delta \sim Re^{-\frac{1}{2}}\mathcal{L} \ll \mathcal{L}$$

'Cross-stream' momentum

$$\rho\left(\hat{v}_{\hat{t}} + \hat{u}\hat{v}_{\hat{x}} + \hat{v}\hat{v}_{\hat{y}}\right) = -\hat{p}_{\hat{y}} + \mu\left(\hat{v}_{\hat{y}\hat{y}} + \hat{v}_{\hat{x}\hat{x}}\right)$$

$$\frac{\rho\mathcal{V}}{\mathcal{T}} \sim \frac{\rho\mathcal{U}\mathcal{V}}{\mathcal{L}} \sim \frac{\rho\mathcal{V}^2}{\delta} \ll \frac{\mathcal{P}}{\delta} \gg \frac{\mu\mathcal{V}}{\delta^2} \gg \frac{\mu\mathcal{V}}{\mathcal{L}^2}$$

Key assumptions

VIP: Pressure constant across layer, i.e.

$$0 = -\hat{p}_{\hat{y}}$$

LIP: Streamwise diffusion negligible, i.e.

$$\bullet_{\hat{y}\hat{y}} \gg \bullet_{\hat{x}\hat{x}}$$

Non-Dimensional Form

Non-dimensionalise

$$\hat{x} \to \mathcal{L}\, x\,, \qquad \hat{y} \to \delta y = Re^{-\frac{1}{2}} \mathcal{L}\, y\,,$$

$$\hat{u} \to \mathcal{U} u\,, \qquad \hat{v} \to \mathcal{V} v = Re^{-\frac{1}{2}} \mathcal{U} v\,,$$

$$\hat{p} \to \rho \mathcal{U}^2\, p\,, \qquad \hat{t} \to \mathcal{T} t = \mathcal{U}^{-1} \mathcal{L}\, t\,.$$

Equations

$$u_t + u u_x + v u_y = -p_x + u_{yy}\,,$$

$$0 = -p_y\,, \qquad u_x + v_y = 0\,.$$

‘Typical’ boundary conditions

$$u = v = 0 \qquad \text{on} \qquad y = 0\,,$$

$$u \to U(x,t) \qquad \text{as} \qquad y \to \infty\,.$$

Steady Flow Past An Aligned Flat Plate

Blasius solution: in terms of similarity variable

$$\eta = y/x^{\frac{1}{2}}$$

Boundary-layer theory (BLT) works . . .

$Re \approx 500$ (Wortmann / Van Dyke)

Steady Flow Past A Circular Cylinder

Boundary-layer 'edge' conditions. As $y \to \infty$

$$\begin{aligned} u &\to U(x) = 2\sin x + \ldots\,, \\ v &\to -U'(x)y + \ldots\,. \end{aligned}$$

'Blowing' velocity. Define

$$v_b(x) = \lim_{y\to\infty} \left(v + U'(x)y\right)\,.$$

Goldstein (1948) singularity.

As $x \to x_c \approx 104.5^\circ$,

$$v_b \sim \frac{\text{const.}}{(x_c - x)^{\frac{1}{2}}} + \ldots\,.$$

A Problem

Flow reversal. At $x = x_c$ the flow near the rigid surface is predicted to reverse.

Negative diffusion. If $u < 0$ somewhere the steady boundary-layer equations, effectively have a region of negative diffusion

$$uu_x + vu_y = UU_x + u_{yy}$$

A real problem. Stewartson (1970) showed that there is no ‘fix-up’ (*inter alia* there is no mechanism for upstream influence). No BLT solution exists for $x > x_c$; moreover the solution for $x < x_c$ is incorrect.

BLT does not always work . . .

Is There A $Re \gg 1$ Solution?

Experimentalist's response:

$$Re \lesssim 41 \quad : \quad \text{steady symmetric}$$

$$Re \gtrsim 41 \quad : \quad \text{unsteady asymmetric}$$

Is 41 large?

Computational scientist's response:

Concurs, but can obtain steady symmetric solutions for $\boldsymbol{Re} \gtrsim \mathbf{41}$ by suppression of instabilities.

Theoretician’s response

Would like to understand reason for ‘singularity’ . . . because it’s there? Hopes it will shed light on the real world.

Because It's There

Unsteady Flow Past A Circular Cylinder

- Impulsively started flow for $Re \gg 1$.
- Boundary layers are initially thin & attached since $\delta \propto t^{\frac{1}{2}}$ when $t \ll 1$:

  $$u_t + uu_x + vu_y = UU_x + u_{yy}$$

  Expect BLT to work for early times.
- Reverse flow starts at the rear stagnation point.
- Reverse flow and separation are not synonymous in unsteady flows.
- Where does 'separation' start?

Prandtl (1932)

Unsteady Separation: ‘Physics’

- At the rear of the cylinder fluid particles are ‘squashed’.

- In Navier-Stokes (NS) or Euler flows it is not possible to squash a particle to zero thickness in one direction and infinite length in another because of the induced pressure gradients.
- In classical BLT

$$p_y = 0$$

- Unsteady separation occurs when the fluid above the squashed particle is ejected from the boundary layer.

Unsteady Separation: 'Mathematics'

Lagrangian co-ordinates: (ξ, η) with

$$\mathbf{x} \equiv \mathbf{x}(\xi, \eta, t) \;, \quad \mathbf{u} \equiv \mathbf{u}(\xi, \eta, t)$$

Kinematics:

$$\left(\frac{\partial x}{\partial t}\right)_{\xi,\eta} = u$$

Momentum equation: independent of y and v

$$\frac{\partial u}{\partial t} = \frac{\partial U}{\partial t} + \left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)\left(\frac{\partial x}{\partial \xi}\frac{\partial}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial}{\partial \xi}\right)u$$

Solve above two equations first.

Mass conservation: hyperbolic equation for y

$$\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \xi} = 1$$

Solve this equation subsequently.

'Shocks': 'shock' formation equivalent to a particle being squashed to zero width.

Van Dommelen & Shen (1980)

Circular Cylinder: Results

Numerical calculations: a singularity develops after about $3/4$ of a diameter movement at

$$t = t_s \approx 1.5\,, \quad x = x_s \approx 111^\circ \neq 104.5^\circ\,.$$

Blowing velocity:

$$v_b\,(x_s, t) \propto \frac{1}{(t_s - t)^{7/4}}$$

Schematic of boundary layer:

Van Dommelen & Shen (1980)

But There Are No NS Singularities . . .

Pressure variation: $p_y = 0$ better written as

$$p_y = O(Re^{-\frac{1}{2}}).$$

Blowing velocity: For $\tau = (t_s - t) \ll 1$

$$v_b = O(\tau^{-\frac{7}{4}}) \quad \text{or} \quad \hat{v}_b = O(\mathcal{U}\, Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}})$$

Induced pressure gradient:

$$\begin{aligned}
\hat{v} &= \phantom{\mathcal{U}U+{}} O(\mathcal{U}\; Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}}) \\
\hat{u} &= \mathcal{U}U + O(\mathcal{U}\; Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}}) \\
\hat{p} &= \phantom{\mathcal{U}U+{}} O(\rho\,\mathcal{U}^2 Re^{-\frac{1}{2}}\, \tau^{-\frac{7}{4}})
\end{aligned}$$

$$p_x = O(Re^{-\frac{1}{2}}\tau^{-\frac{13}{4}})$$

Elliott *et al.* (1983)

The Interaction Problem

BL: Consider induced pressure and acceleration in $x - x_s = O(\tau^{\frac{3}{2}})$ region

$$\begin{aligned} p_x &= O(Re^{-\frac{1}{2}}\tau^{-\frac{13}{4}}) \\ u_t = x_{tt} &= O(\tau^{-\frac{1}{2}}) \end{aligned}$$

Interaction: when $\tau = O(Re^{-\frac{2}{11}})$ need to solve a quadruple-deck problem.

'Validity': for distinct regions need $Re^{\frac{1}{22}} \gg 1$.

'Numerical solution': terminates in another singularity — need to rescale again . . .

Cassel *et al.* (1996)

The Next Interaction Problem

(Rescaled)^2 problem: model version on this even shorter timescale leads to [yet] another singularity.

Li *et al.* (1998)

Has something gone wrong? Not really . . . high Reynolds number flows are 'turbulent'!

Have excited the instabilities that lead to small scale turbulent structure.

What about times $t - t_s = O(1)$?

Good news / bad news. BLT predicts both unsteady separation and what effects then come in . . . it does not provide a long-time predictor.

But ‘we’ hoped that they were laminar.

Interesting question . . . maybe by introducing clever averaging/multiple-time-scales . . . but I would not bet on it.

Well That's Sorted . . . Or Is It?

$\{$ BLT $\}$ = $\{$ $\boldsymbol{Re} \gg 1$ NS $\}$? Sometimes . . . but maybe not always?

Numerical results: $\boldsymbol{Re} \gg 1$ NS calculations, that should 'tend' to the unsteady BLT solution as $\boldsymbol{Re}$ increases, show instabilities *before* the separation time.

Brinckman & Walker (200?)

Rayleigh Instabilities

Inflection points. If $p_x > 0$ the velocity profile $u(x,y,t)$ has an inflection point — since if there is no velocity overshoot

$$u_{yy} < 0 \quad \text{as} \quad y \to \infty$$

while on $y = 0$

$$u_t + uu_x + vu_y = -p_x + u_{yy} \quad \Rightarrow \quad u_{yy} > 0$$

Instabilities. On $Re^{-\frac{1}{2}}$ lengthscales, Rayleigh instabilities exist (Tollmien 1936)

$$\text{amplitude} \ \propto \ \exp(Re^{\frac{1}{2}}\,\beta\,t)$$

Short waves grow very fast

— but is there anything to grow?

Decay Of Spectrum: Speculation

Analytically continue the classical BLT solution, $u(x, y, t)$, into the complex x–plane.

Hypothesise that there are singularities in the complex x–plane that intersect the real x–axis at $t = t_s$ (cf. vortex sheet).

Spectrum: if $\tilde{u}(k, y, t)$ is the k^{th} term of the Fourier series of $u(x, y, t)$, then

$$\tilde{u}(k, y, t) \propto \exp(-\alpha k) \quad \text{as} \quad |k| \to \infty$$

Hypothesise that exponential decay holds down to Rayleigh scale

$$k = O(Re^{\frac{1}{2}})$$

Growth Of Instabilities

Exponentially small? Consider the growth of the NS high wavenumber spectrum by Rayleigh instability

$$\widetilde{u}(Re^{\frac{1}{2}}, y, t) \propto \exp(-Re^{\frac{1}{2}}(\alpha - \beta\, t))$$

$O(1)$ when $t = O(1)$! Not inconsistent with numerical results! Race between Rayleigh instability and development of singularity.

Comments

- No disturbance introduced on short Rayleigh scale — self-induced nonlinear effects sufficient to fill out spectrum.
- No hint of problem from BLT, i.e. $k = O(1)$ asymptotics.
- Need to worry about terms that are initially exponentially small — or non-existent.

A Model Problem

Kuramoto-Sivashinsky (KS) equation:

$$u_t + uu_x = -\epsilon u_{xx} - \delta u_{xxxx}$$

$$u = \sin x \quad \text{at} \quad t = 0$$

$\delta = \epsilon = 0$: analytic solution, $u_0(x,t)$, develops singularity at $t = 1$. For $t > 0$ and $k \gg 1$

$$\widetilde{u}_0(k,t) \propto \exp(-\alpha(t)k)$$

$0 < \delta \ll \epsilon \ll 1$:

KS well-posed, but $\exists$ rapidly growing high–wavenumber instabilities.

E.g. when $k^2 = \epsilon/2\delta \gg 1$

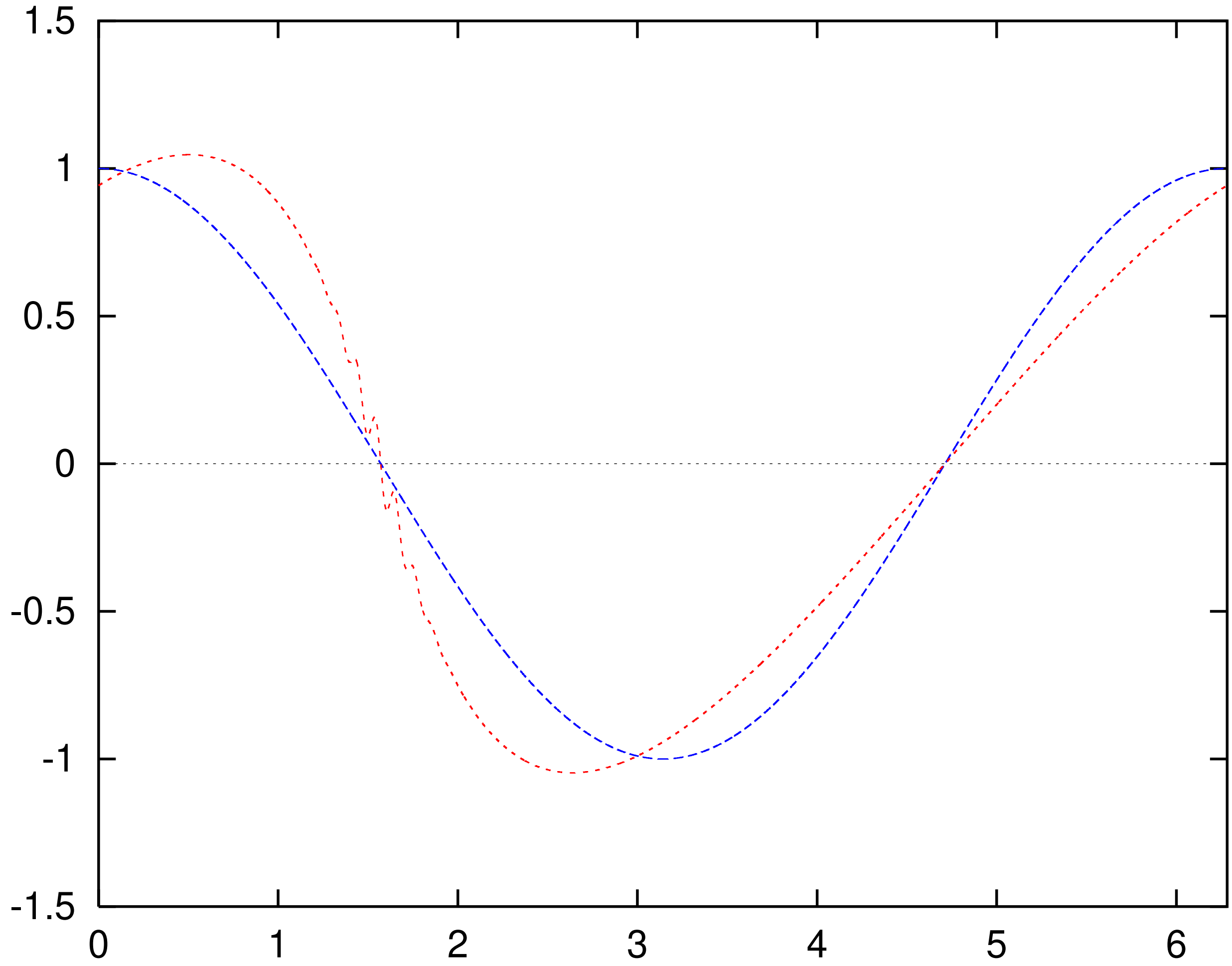
$$\text{amplitude} \propto \exp(\epsilon^2\, t/2\delta)\,.$$

$u_0(x,t)$ ‘infected’ when

$$t \sim \delta^{\frac{1}{2}}\epsilon^{-\frac{3}{2}}?$$

$\delta = O(\epsilon^3)$: instability before singularity.

Numerical Solution To KS Equation



Numerical solution with $\epsilon = 0.1$, $\delta = 0.00001$ and $u = \sin x$ at $t = 0$.

— — — : solution at $t = 0$

— — — : solution at $t = 0.46$

If $\epsilon = \delta = 0$ a shock singularity forms at $t = 1$.

Comments On KS Problem

Heuristic argument seems to be supported by numerical experiment.

Single mode initial condition emphasises that nonlinear interactions generate the 'exponentially small' higher modes.

Naïve asymptotics can lead to the wrong 'solution'.

Possible Problems With Classical BLT

Leading-order problem can lead to a succession of singularities forcing consideration of extremely short time scales.

Terms that start exponentially small might grow to alter the leading-order solution (cf. Siegel, Tanveer & Dai 1996).

If You Can't Beat 'Em, Join 'Em!

Laminar $Re \gg 1$ analysis problematic.

- Sometimes successful, e.g. Blasius solution at $Re = 500$.
- Sometimes not, e.g. flow past a circular cylinder where instabilities create problems.

Study instabilities instead, e.g. for

- steady flows where $Re \gg 1$ asymptotics are a good approximation (e.g. flat plate);
- starting flows, since quiescent states are stable.

‘Parallel’ Flows: Orr-Sommerfeld Theory

Almost parallel $\rightarrow$ parallel: i.e. assume

$$\mathbf{u} \equiv (U(y), 0, 0) = \mathbf{U}$$

Linear instability: normal-mode analysis

$$\mathbf{u} = \mathbf{U} + \epsilon\tilde{\mathbf{u}}(y)\exp(\mathrm{i}\alpha x + \mathrm{i}\beta z - \mathrm{i}\alpha c t) + \ldots$$

Dispersion relation and neutral curve:

$$F(\alpha, \beta, c; Re) = 0$$

Sometimes it works: 2D TS waves (flat plate)

Sometimes it doesn’t: Görtler rolls and rotating disk instability.

Orr-Sommerfeld (OS) Theory: Drawbacks

Mathematically inconsistent:

- ‘almost’ parallel $\Rightarrow Re \gg 1$.
- $Re = O(1)$ in OS dispersion relation.

Problems: not clear how to deal with

- non-parallel effects;
- non-linear effects.

Solution? Assume $Re \gg 1$ throughout.

Bad news: difficult (if not impossible) to study fastest-growing disturbance.

Good news: *sometimes* not necessary.

Tollmien-Schlichting Lower Branch

Triple-deck Analysis

Upper: inviscid, irrotational, elliptic

$$\nabla^2 \widetilde{p} = 0$$

Middle: inviscid, rotational, displacement.

Lower: viscous critical layer

$$u = O(c)$$

Tollmien (1929), Lin (1945), Smith (1979).

Triple Deck: Analysis

Expand in powers of $Re^{-\frac{1}{8}}$:

- matched asymptotics in y
- multi-scale/WKB[JLG] in x and/or t.

Linear TS lower-branch ‘works’

Smith 1979, Healey 1995

Asymptotic Nonlinear Instablilty

A plethora of possibilities:

Good news — employment for academics.

Bad news — which, if any, work?

Different types of modes: e.g. Rayleigh, Klebanoff, Tollmien-Schlichting.

Different sources of instability: e.g. 2D / 3D, localised / global, controlled / noise, linear / nonlinear, internal / external.

Different types of analysis: many studies of almost neutral modes (modulated wavetrains), but other possibilities (e.g. for Klebanoff modes).

Wave/Mean-Flow Effects

Two oblique modes: steady mean flow

$$\left(\mathrm{e}^{\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{-\mathrm{i}\omega t}\right)\left(\mathrm{e}^{-\mathrm{i}\alpha x}\mathrm{e}^{\mathrm{i}\beta z}\mathrm{e}^{\mathrm{i}\omega t}\right) = \mathrm{e}^{2\mathrm{i}\beta z}$$

mode 1 **c.c. of mode 2**

Disjoint scales. If spanwise, β^{-1}, and 'slow' streamwise, $\mathcal{L}$, scales disjoint ($\beta\mathcal{L} \ll 1$), modest spanwise motions generate large streamwise mean flows:

$$u_x + v_y + w_z = 0$$

$$\frac{\mathcal{U}_m}{\mathcal{L}} \qquad \beta\mathcal{W}_m$$

$$\Rightarrow \mathcal{U}_m \sim \beta\mathcal{L}\mathcal{W}_m \gg \mathcal{W}_m$$

Critical Layers/Levels

Weakly nonlinear analysis: at leading order

$$u_t + uu_x \to \mathrm{i}\alpha(u - c)u$$

Critical layer where $u = c$, say y_c.

Singularity in linear inviscid solution — in 3D

$$\text{perturbation amplitude} \propto \frac{1}{y - y_c}$$

Singularity can be smoothed out by

- viscosity
- time-dependent effects
- nonlinearity
- $\ldots$

Nonlinear effects 'largest' in critical layer: study nonlinear interactions of modes with the same phase speed c.

Pitfalls

Complicated analysis: e.g. resonant triad interaction requires a 'septuple-deck' structure.

Singularities in modulation amplitude:

- look sexy, but do they describe transition?
- often lead to a succession of increasingly short timescales (cf. unsteady separation).

Agreement with experiment: rarely correlates with mathematically satisfying analysis!

Receptivity: How A Disturbance Penetrates A Shear Layer

Sound waves & 'rivet'. If 'rivet' size matches triple-deck lengthscale (or ∃ a curvature discontinuity), TS waves excited.

Sound waves & gust. Suppose

$$u = U + \epsilon_s e^{i\omega_s t - i\alpha_s x} + \epsilon_g e^{i\omega_g t - i\alpha_g x} + \dots$$

where $\alpha_s = M\omega_s/(1+M)$ and $\alpha_g = \omega_g$.

If $\alpha_s - \alpha_g = O(Re^{\frac{3}{8}})$ and $\omega_s - \omega_g = O(Re^{\frac{1}{4}})$, quadratic interactions in the 'upper deck' between sound waves and gusts generate TS waves. Applicable to broad-band spectrum.

Ruban (1984), Goldstein (1985), Wu (2000) . . .

TS Resonant Triad Instability

Original analysis. BL-resonant-triad interaction proposed by Raetz and Craik — used OS approach.

Asymptotic analysis. Put on firm asymptotic footing by Goldstein, Lee, Wu, & Mankbadi.

'First stage' amplitude equations:

2D mode

$$\frac{dA}{dX} = \sigma_a A$$

3D mode

$$\frac{dB}{dX} = \sigma_b B + qAB^* + hB\int_0^\infty |B(X-\zeta)|^2 d\zeta$$

- Both resonant triad and wave/mean-flow interactions.
- Super-exponential growth.

Succession of rescalings subsequently necessary on shorter & shorter timescales.

Quantitative agreement between theory and experiment lacking!

Successes

Receptivity.

Resonant-triad instability: qualitative . . . but not quantitative . . . Craik, Goldstein, Lee, Wu, Mankbadi . . .

Nonlinear roll-up of a shear layer: Hultgren & Goldstein (1989).

Linear Görtler instability: Hall (1983, . . .).

Buran: . . .

Ask the CIA/NSA, fKGB, . . . , maybe NASA.

Laminar Boundary-Layer Theory: A Paradox.

- Laminar BLT assumes $Re \gg 1$.
- Almost all flows are turbulent if $Re \gg 1$.
- To obtain laminar solutions need to suppress instabilities. Sometimes possible:
  - Blasius flat-plate solution;
  - receptivity;
  - start of unsteady separation?

  Sometimes not:
  - ‘medium-term’ unsteady separation.
- Analysis often results in a succession of problems with smaller & smaller length/time scales.
- Beware of exponentially small terms.

Laminar Boundary-Layer Theory: A 20th Century Paradox ?

No. BLT still good for

- explaining fundamental mechanisms;
- obtaining scalings.

Yes.

- For quantitative agreement with experiment BLT outgunned by CFD.

Laminar Boundary-Layer Theory: a 20th Century Paradox.

The USA:
a 20th Century Superpower?