1B Methods – Example Sheet 4

Please email me with any comments, particularly if you spot an error. Problems marked with an asterisk (∗) are optional; only attempt them if you have time.

1. (a) Find the characteristic curves of $\partial_x u + \partial_y u = 0$, and find u if $u(0, y) = y^3$.

   (b) Solve $y\partial_x u + x\partial_y u = 0$ subject to $u(0, y) = e^{-y^2}$. In which region of the plane is the solution uniquely determined?

   (c) Find u such that $\partial_x u + \partial_y u + u = e^{x+2y}$, and $u(x, 0) = 0$.

2. The *backward* diffusion equation may be defined as

$$\partial_t u + \partial_x^2 u = 0\,.$$

   Consider a domain $0 < x < \pi$, with $u(0, t) = 0 = u(\pi, t)$, and $u(x, 0) = U(x)$. By using the method of separation of variables, show that the problem is not well-posed. [*It may be helpful to scale the eigenfunctions you calculate similarly to the example in the lectures.*]

3. (a) Determine the regions of the plane where Tricomi's equation,

$$\partial_x^2 u + x\,\partial_y^2 u = 0\,,$$

   is of elliptic, parabolic and hyperbolic types. Derive its characteristics and canonical form in the hyperbolic region.

   (b) Reduce the equation

$$\partial_x^2 u + y\,\partial_y^2 u + \frac{1}{2}\partial_y u = 0$$

   to the simple canonical form $\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$ in its hyperbolic region, and hence show that

$$u = f(x + 2\sqrt{-y}) + g(x - 2\sqrt{-y}),$$

   where f and g are arbitrary functions.

4. Consider a Dirichlet Green's function $G(\mathbf{x}; \mathbf{y})$ for the Laplacian defined inside some compact region $\Omega \subset \mathbb{R}^3$. By using Green's second identity, show that $G(\mathbf{x}; \mathbf{y}) = G(\mathbf{y}; \mathbf{x})$ for all $\mathbf{x} \in \Omega$ with $\mathbf{x} \neq \mathbf{y}$.

5. Suppose u is harmonic inside a compact domain $\Omega \subset \mathbb{R}^2$, with boundary $\partial\Omega$. Show that

$$u(\mathbf{y}) = \frac{1}{2\pi}\oint_{\partial\Omega} \left[u(\mathbf{x})\,\hat{\mathbf{n}}\cdot(\nabla \log|\mathbf{x}-\mathbf{y}|) \;-\; \log|\mathbf{x}-\mathbf{y}|\;\hat{\mathbf{n}}\cdot(\nabla u(\mathbf{x}))\right] dl\,,$$

   where dl is an arc element of $\partial\Omega$, with $\mathbf{x} \in \partial\Omega$ and $\mathbf{y} \in \Omega$.

6. Let $\Omega = \{(x, y) \in \mathbb{R}^2 \,|\, y \geq 0\}$ and suppose $\phi : \Omega \to \mathbb{R}$ solves $\nabla^2\phi = 0$ inside Ω, subject to the boundary conditions $\phi(x, 0) = f(x)$ for some known function f, and $\lim_{y\to\infty} \phi = 0$.

   (a) Find the Green's function for this problem.

   (b) Hence show that the solution is given by Poisson's integral formula:

$$\psi(x, y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2 + y^2} d\xi\,.$$

   (c) Derive the same result by taking Fourier transforms with respect to x (assuming all transforms exist).

   (d) Find (in closed form) and sketch the solution for various $y > 0$ when $f(x) = \psi_0$, $|x| < a$, and $f(x) = 0$ otherwise. Sketch the solution along $x = \pm a$.

7. A string is at rest for all $t < 0$ and under tension. At $t = 0$ the string receives an instantaneous transverse blow which imparts an initial velocity $V\,\delta(x - x_0)$, where V is a constant and $x_0 > 0$. Find the position of the string for all $t > 0$ if:

   (a) The string is infinitely long and free to osillate.

   (b) The string is pinned down at $x = 0$.

   Compare the two solutions.

8. Let $\Omega = \{(x, t) \in \mathbb{R}^2_{\geq}\}$ (*i.e.*, x and t non-negative). Suppose θ solves the forced heat equation

$$\partial_t\theta - D\partial_x^2\theta = f(x, t)$$

   inside Ω, and that $\theta(0, t) = h(t)$ and $\theta(x, 0) = \Theta(x)$. By considering $V(x, t) = \theta(x, t) - h(t)$, use the method of images to obtain the general solution.

9*. Show that the Dirichlet Green's function for the Laplacian for the *interior* of a spherical domain of radius a is

$$G(\mathbf{x}; \mathbf{y}) = -\frac{1}{4\pi|\mathbf{y}|}\left[\frac{|\mathbf{y}|}{|\mathbf{x} - \mathbf{y}|} - \frac{a}{|\mathbf{x} - \mathbf{y}^*|}\right]\,,$$

   where $\mathbf{y}^* := a^2\mathbf{y}\,/\,|\mathbf{y}|^2$. Also derive the Dirichlet Green's function for the Laplacian in the region *exterior* to this sphere.