

Example Sheet 4

1. Radial oscillations of a star

Show that purely radial (i.e. spherically symmetric) oscillations of a spherical star satisfy the Sturm–Liouville equation

$$\frac{d}{dr} \left[\frac{\gamma p}{r^2} \frac{d}{dr} (r^2 \xi_r) \right] - \frac{4}{r} \frac{dp}{dr} \xi_r + \rho \omega^2 \xi_r = 0.$$

How should ξ_r behave near the centre of the star and near the surface $r = R$ at which $p = 0$?

Show that the associated variational principle can be written in the equivalent forms

$$\begin{aligned} \omega^2 \int_0^R \rho |\xi_r|^2 r^2 dr &= \int_0^R \left[\frac{\gamma p}{r^2} \left| \frac{d}{dr} (r^2 \xi_r) \right|^2 + 4r \frac{dp}{dr} |\xi_r|^2 \right] dr \\ &= \int_0^R \left[\gamma p r^4 \left| \frac{d}{dr} \left(\frac{\xi_r}{r} \right) \right|^2 + (4 - 3\gamma)r \frac{dp}{dr} |\xi_r|^2 \right] dr, \end{aligned}$$

where γ is assumed to be independent of r . Deduce that the star is unstable to purely radial perturbations if and only if $\gamma < 4/3$. Why does it not follow from the first form of the variational principle that the star is unstable for all values of γ ?

Can you reach the same conclusion using only the virial theorem?

2. Waves in an isothermal atmosphere

Show that linear waves of frequency ω and horizontal wavenumber k_h in a plane-parallel isothermal atmosphere satisfy the equation

$$\frac{d^2 \xi_z}{dz^2} - \frac{1}{H} \frac{d \xi_z}{dz} + \frac{(\gamma - 1)}{\gamma^2 H^2} \xi_z + (\omega^2 - N^2) \left(\frac{1}{v_s^2} - \frac{k_h^2}{\omega^2} \right) \xi_z = 0,$$

where H is the isothermal scale-height, N is the Brunt–Väisälä frequency and v_s is the adiabatic sound speed.

Consider solutions of the vertically wavelike form

$$\xi_z \propto e^{z/2H} \exp(ik_z z),$$

where k_z is real, so that the wave energy density (proportional to $\rho |\boldsymbol{\xi}|^2$) is independent of z . Obtain the dispersion relation connecting ω and \mathbf{k} . Assuming that $N^2 > 0$, show that propagating waves exist in the limits of high and low frequencies, for which

$$\omega^2 \approx v_s^2 k^2 \quad (\text{acoustic waves}) \quad \text{and} \quad \omega^2 \approx \frac{N^2 k_h^2}{k^2} \quad (\text{gravity waves})$$

respectively. Show that the minimum frequency at which acoustic waves propagate is $v_s/2H$.

Explain why the linear approximation must break down above some height in the atmosphere.

3. Magnetic buoyancy instabilities

A perfect gas forms a static atmosphere in a uniform gravitational field $-g \mathbf{e}_z$, where (x, y, z) are Cartesian coordinates. A horizontal magnetic field $B(z) \mathbf{e}_y$ is also present.

Derive the linearized equations governing small displacements of the form

$$\text{Re} [\boldsymbol{\xi}(z) \exp(-i\omega t + ik_x x + ik_y y)] ,$$

where k_x and k_y are real horizontal wavenumbers, and show that

$$\omega^2 \int_a^b \rho |\boldsymbol{\xi}|^2 dz = [\xi_z^* \delta \Pi]_a^b + \int_a^b \left(\frac{|\delta \Pi|^2}{\gamma p + \frac{B^2}{\mu_0}} - \frac{|\rho g \xi_z + \frac{B^2}{\mu_0} i k_y \xi_y|^2}{\gamma p + \frac{B^2}{\mu_0}} + \frac{B^2}{\mu_0} k_y^2 |\boldsymbol{\xi}|^2 - g \frac{d\rho}{dz} |\xi_z|^2 \right) dz,$$

where $z = a$ and $z = b$ are the lower and upper boundaries of the atmosphere, and $\delta \Pi$ is the Eulerian perturbation of total pressure. (Self-gravitation may be neglected.)

You may assume that the atmosphere is unstable if and only if the integral on the right-hand side can be made negative by a trial displacement $\boldsymbol{\xi}$ satisfying the boundary conditions, which are such that $[\xi_z^* \delta \Pi]_a^b = 0$. You may also assume that the horizontal wavenumbers are unconstrained. Explain why the integral can be minimized with respect to ξ_x by letting $\xi_x \rightarrow 0$ and $k_x \rightarrow \infty$ in such a way that $\delta \Pi = 0$.

Hence show that the atmosphere is unstable to disturbances with $k_y = 0$ if and only if

$$-\frac{d \ln \rho}{dz} < \frac{\rho g}{\gamma p + \frac{B^2}{\mu_0}}$$

at some point.

Assuming that this condition is not satisfied anywhere, show further that the atmosphere is unstable to disturbances with $k_y \neq 0$ if and only if

$$-\frac{d \ln \rho}{dz} < \frac{\rho g}{\gamma p}$$

at some point.

How does these stability criteria compare with the hydrodynamic stability criterion $N^2 < 0$?

4. Waves in a rotating fluid

Write down the equations of ideal gas dynamics in cylindrical polar coordinates (r, ϕ, z) , assuming axisymmetry. Consider a steady, axisymmetric basic state in uniform rotation, with density $\rho(r, z)$, pressure $p(r, z)$ and velocity $\mathbf{u} = r\Omega \mathbf{e}_\phi$. Determine the linearized equations governing axisymmetric perturbations of the form

$$\text{Re} [\delta\rho(r, z) e^{-i\omega t}] ,$$

etc. If the basic state is adiabatically stratified (i.e. $s = \text{constant}$) and self-gravity may be neglected, show that the linearized equations reduce to

$$\begin{aligned} -i\omega \delta u_r - 2\Omega \delta u_\phi &= -\frac{\partial W}{\partial r} , \\ -i\omega \delta u_\phi + 2\Omega \delta u_r &= 0 , \\ -i\omega \delta u_z &= -\frac{\partial W}{\partial z} , \\ -i\omega W + \frac{v_s^2}{\rho} \left[\frac{1}{r} \frac{\partial}{\partial r} (r\rho \delta u_r) + \frac{\partial}{\partial z} (\rho \delta u_z) \right] &= 0 , \end{aligned}$$

where $W = \delta p / \rho$.

Eliminate $\delta \mathbf{u}$ to obtain a second-order partial differential equation for W . Is the equation of elliptic or hyperbolic type? What are the relevant solutions of this equation if the fluid has uniform density and fills a cylindrical container $\{r < a, 0 < z < H\}$ with rigid boundaries?

Please send any comments and corrections to gio10@cam.ac.uk