Part III Magnetohydrodynamics
Michaelmas 2015

Examples I

1. Using the approximations employed in lectures, show that comparing the laboratory frame with a moving frame with speed $U \ll c$, the magnetic fields in the two frames can be taken as equal, while the 'current' $\rho_E U$ can be neglected in comparison with the electric current j.

2. A uniform magnetic field $\mathbf{B}_0 = (0, 0, B_0)$ in cartesian coordinates (x, y, z) is perturbed by the velocity field $\mathbf{u} = (U_0 \,\mathrm{sech}^2 kz \tanh kz, 0, 0)$. Find

   (i) the form of the initial perturbation to $\mathbf{B}$, neglecting diffusion

   (ii) the final steady state.

   What is the characteristic time for diffusion to become important?

3. Now repeat the calculation (i) above when $\mathbf{B}_0$ is replaced by $\mathbf{B}_1 = (0, 0, B_1 \cos \ell x)$, and $R_m = U_0/k\eta$ is large. By estimating the relative size of the diffusion term to the other terms in the equation, when the diffusionless solution is used, give an estimate of the time that elapses before diffusion becomes important in this case?

4. Consider the action of an axisymmetric stagnation point flow $\mathbf{u} = (-Uh(t)s/2d, 0, Uh(t)z/d)$ (in cylindrical polar coordinates (s, ϕ, z)) on a magnetic field $\mathbf{B}$ that only has a z-component $B(s, t)$. Write down the equation satisfied by B, verify that no other components of B are induced by the flow, and show that there is a solution of the form

$$B(r, t) = g(t) \exp(-f(t)s^2)),$$

   Verify from the equations that the total magnetic flux $2\pi \int_0^\infty sB\, ds$ is conserved. and hence show that $f(t) \propto g(t)^2$; taking $f(t) = g(t)^2$ find the equation satisfied by g. Without writing down an explicit solution show that if $h(t)$ is periodic in time and if there is a solution in which g is positive and finite at all times then the time average of h must be positive.

5. Consider the action of a stagnation point flow $\mathbf{u} = A\mathbf{x}$, where the trace of the matrix A is zero.

   Show that the induction equation has a solution of the form $\mathbf{B} = \hat{\mathbf{B}}(t)e^{i\mathbf{k}(t)\cdot\mathbf{x}}$. Write down the equations for $\mathbf{B}$, $\mathbf{k}$,and use them to describe the subsequent evolution of $\mathbf{B}$.

6. A solid circular cylinder of radius a rotates about the z-axis with angular velocity Ω. Far from the cylinder there is a uniform magnetic field B in the x direction. Outside the cylinder the material is insulating, while the cylinder itself is conducting with magnetic diffusivity η. The magnetic Reynolds number $R_m = \Omega a^2/\eta$ is very large.

   Assuming that the resulting steady magnetic field is two-dimensional (no z component, independent of z), and that $\mathbf{B} = \nabla \times (0, 0, A)$ in cylindrical polar coordinates (r, ϕ, z), show that $A = \mathbf{Re}\hat{A}(r)e^{i\phi}$ and that

$$i\Omega\hat{A} = \eta\left(\hat{A}_{rr} + \frac{1}{r}\hat{A}_r - \frac{1}{r^2}\hat{A}\right), \ \ r < a; \quad \hat{A}_{rr}+\frac{1}{r}\hat{A}_r-\frac{1}{r^2}\hat{A} = 0, \ r > a; \quad \hat{A} \sim Br, \ \ r \to \infty.$$

   Give the boundary conditions satisfied by $\hat{A}$ at $r = a$. Show that when R_m is large the magnetic field in the cylinder is confined to a thin boundary layer near $r = a$, of thickness $aR_m^{-\frac{1}{2}}$.

7. Alfvén waves. Consider perturbations $\mathbf{u}(\mathbf{x}, t)$, $\mathbf{b}(\mathbf{x}, t)$ to a state with no velocity and uniform magnetic field $\mathbf{B}$. For an incompressible fluid the equations of motion and induction are

$$\begin{aligned}(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)\mathbf{u} &= -\nabla p + (\rho\mu_0)^{-1}\nabla \times \mathbf{b} \times (\mathbf{B} + \mathbf{b}) + \nu\nabla^2\mathbf{u} \\ (\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla)\mathbf{b} &= (\mathbf{B} + \mathbf{b}) \cdot \nabla\mathbf{b} + \eta\nabla^2\mathbf{b} \\ \nabla \cdot \mathbf{u} &= \nabla \cdot \mathbf{b} = 0\end{aligned}$$

   (i) Linearize the equations for small $\mathbf{u}, \mathbf{b}$ and seek spatially periodic solutions of the form $\mathbf{u}$, $\mathbf{b} \propto \exp(\sigma t + i\mathbf{k} \cdot \mathbf{x})$. Find the decay rate σ of such disturbances, and show that when the *Lundqvist number* $\mathcal{L} = |\mathbf{B}|/(\sqrt{\mu_0\rho\nu\eta}|\mathbf{k}|)$ is sufficiently large then σ is complex, corresponding to (damped) waves. Find approximations to the roots when the *Magnetic Prandtl number* $P_m = \nu/\eta$ is very large or very small.

   (ii) Now ignore the diffusion terms, but retain the nonlinear terms. Show that the equations have exact solutions in the form of waves that travel at the *Alfvén velocity* $\mathbf{V} = \mathbf{B}/\sqrt{\mu_0\rho}$.