Part III Magnetohydrodynamics
Michaelmas 2015

Examples II

1. By expanding the expression derive the three components of the poloidal magnetic field $\mathbf{B}_P = \nabla \times \nabla \times (P\mathbf{r})$, and thus show that $\nabla \times \mathbf{B}_P = \nabla \times (-(\nabla^2 P)\mathbf{r})$.

2. Consider an arbitrary poloidal field $\mathbf{P}$ and an arbitrary toroidal field $\mathbf{Q}$. Show that for any spherical surface S_0 with $r =$ const. we must have

$$\int_{S_0} \mathbf{P} \cdot \mathbf{Q}\, ds = 0$$

3. Consider the situation envisaged in lectures for the toroidal theorem for which $\mathbf{u}\cdot\mathbf{r} \equiv 0$, and $\mathbf{B} = \mathbf{B}_T = \nabla \times (T\mathbf{r})$. Show that T obeys the equation

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \eta \nabla^2 T$$

in $r < a$, with $T = 0$ on $r = a$. Deduce that $\int_{r\le a} T^2\, dV \to 0$ as $t \to \infty$.

(*) Does this imply that $\int_{r\le a} |\mathbf{B}_T|^2\, dV \to 0$?

4. For a solenoidal axisymmetric flow $\mathbf{u}$ in a sphere V of radius a surrounded by insulator, and axisymmetric field $\mathbf{B} = B\hat{\boldsymbol{\phi}} + \nabla \times (A\hat{\boldsymbol{\phi}})$ we know that A in V obeys the equation

$$\frac{\partial A}{\partial t} + \frac{1}{s}\mathbf{u} \cdot \nabla (sA) = \eta (\nabla^2 A - \frac{1}{s^2} A).$$

Write down the equation satisfied by A in the insulating region. Show, stating carefully what assumptions you are making, that $A \to 0$ as $t \to \infty$.

5. Consider the mean emf due to a steady solenoidal velocity field $\mathbf{u}(\mathbf{x})$ at small magnetic Reynolds number R_m. The velocity field is monochromatic, so that $\nabla^2 \mathbf{u} = -\mathbf{u}$. Consider the induction equation in the form

$$0 = \overline{\mathbf{B}} \cdot \nabla \mathbf{u} + \nabla \times (\mathbf{u} \times \mathbf{B}') + \frac{1}{R_m} \nabla^2 \mathbf{B}',$$

where $\overline{\mathbf{B}}$ is a constant vector, and $\mathbf{B}'$ is the induced field. Show that the mean emf $\mathcal{E} = \overline{\mathbf{u} \times \mathbf{B}'}$, where the overbar denotes an average over all space, can be written in the form

$$\mathcal{E} = R_m \mathcal{E}^{(1)} + R_m^2 \mathcal{E}^{(2)} + \ldots = R_m \overline{\mathbf{u} \times \overline{\mathbf{B}} \cdot \nabla \mathbf{u}} + R_m^2 \overline{\mathbf{u} \times \nabla \times (\mathbf{u} \times \overline{\mathbf{B}} \cdot \nabla \mathbf{u})} + \ldots$$

Show, without using Fourier decomposition, that if $\mathcal{E}_i^{(p)}$ can be written $\alpha_{ij}^{(p)} \overline{B}_j$, then $\boldsymbol{\alpha}^{(1)}$ is symmetric.

Derive the result

$$\mathcal{E}_i^{(2)} = -\overline{\frac{\partial u_j}{\partial x_i} (\mathbf{u} \times \overline{\mathbf{B}} \cdot \nabla \mathbf{u})_j}\ .$$

6. Prove an exact result for the α-effect, namely that if $(\overline{\mathbf{u} \times \mathbf{B}'})_i = \mathcal{E}_i = \alpha_{ij}\overline{\mathbf{B}}_j$, then

$$\mathcal{E} \cdot \overline{\mathbf{B}} = \alpha_{ij}\overline{\mathbf{B}}_i\overline{\mathbf{B}}_j = -\frac{1}{\eta}\overline{\mathbf{B}' \cdot \nabla \times \mathbf{B}'},$$

for a mean field dynamo in a statistically steady state, stating carefully any assumptions you make. Verify that in the First Order Smoothing limit the expression agress with the result derived in lectures. *[Hint: consider the equation for the vector potential].*

7. Find conditions for the existence of a steady α^2-dynamo with uniform α satisfying

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\alpha \mathbf{B}) + \eta \nabla^2 \mathbf{B}$$

in a conducting sphere of radius a surrounded by insulator. (Use the poloidal-toroidal decomposition as for the free decay modes in lectures.) Sketch the lines of poloidal field and contours of toroidsl field in the case of an axisymmetric solution (which corresponds to the lowest value of $|\alpha|$.) Why is a model with constant α unsatisfactory as a consequence of the effect of rotation in small-scale flows?

8. A nonlinear model of Parker dynamo waves, incorporating the effects of induced zonal flow, takes the form of four ODEs:

$$\begin{aligned}
\dot{A} &= DB - A, \quad \dot{B} = iA(1 + U_0) - B + iU_2A^*, \\
\dot{U}_0 &= \frac{i}{2}(A^*B - AB^*) - \nu_0 U_0, \\
\dot{U}_2 &= iAB - \nu_2 U_2.
\end{aligned}$$

Here A,B represent poloidal and toroidal fields, respectively, proportional to e^{ikx}, while U_0 (real) represents the zonal flow perturbation independent of x, and, U_2 the zonal flow perturbation proportional to e^{2ikx}.

(i) Verify that the nonlinear terms make no contribution to the rate of change of total "energy" $|A|^2 + |B|^2 + U_0^2 + |U_2|^2$.

(ii) Show that travelling wave solutions, with $A, B \propto e^{i\Omega t}$, U_0 steady and $U_2 \propto e^{2i\Omega t}$, can be found, and write down equations giving (in parametric form) the relation between D, Ω and $|A|^2$. Find approximate formulae for D and Ω when $|A|^2$ is (a) very small and (b) very large.