

Example Sheet 1

1. *Revision of Keplerian orbits*

The equation of motion of a test particle in the gravitational field of a point mass M is

$$\ddot{\mathbf{r}} = -\frac{GM\mathbf{r}}{|\mathbf{r}|^3}.$$

Show that the motion is confined to a plane containing the central mass. Introduce polar coordinates (r, ϕ) in the plane and deduce that

$$\ddot{r} - r\dot{\phi}^2 = -\frac{GM}{r^2},$$

$$r^2\dot{\phi} = h = \text{constant}.$$

Find an equation for the shape $r(\phi)$ of the orbit and show that the general solution is

$$r = \frac{\lambda}{1 + e \cos(\phi - \omega)},$$

where e and ω are arbitrary constants, and $\lambda = h^2/GM$. Sketch the orbit for the cases $e = 0$, $0 < e < 1$, $e = 1$ and $e > 1$.

Show that the orbit of least energy for a given angular momentum is a circular orbit.

2. *Precession in Newtonian dynamics*

- (i) Given an axisymmetric gravitational potential $\Phi(r, z)$ for which $z = 0$ is a plane of symmetry, explain how to determine the angular velocity $\Omega(r)$ of circular orbits in the plane $z = 0$, as well as the epicyclic frequency $\kappa(r)$ and the vertical frequency $\Omega_z(r)$.
- (ii) In regions of space where Φ satisfies Laplace's equation (i.e. away from the important sources of the gravitational field), show that the three frequencies are related by

$$\kappa^2 + \Omega_z^2 = 2\Omega^2. \tag{1}$$

What does this relation imply for the directions of apsidal and nodal precession?

- (iii) The exterior gravitational potential of a rotating planet or star can be written in the form

$$\Phi = -\frac{GM}{R} \left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R_e}{R} \right)^n P_n(\cos \theta) \right],$$

where M is the mass of the body, R_e is its equatorial radius, (R, θ, ϕ) are *spherical* polar coordinates, J_n is a dimensionless coefficient and P_n is the Legendre polynomial of degree n .

Explain why this form is reasonable, and why it is reasonable that $J_n = 0$ for odd values of n .

Find expressions for $\Omega(r)^2$, $\kappa(r)^2$ and $\Omega_z(r)^2$. Verify that equation (1) is satisfied. Show that the leading approximations for the apsidal and nodal precession rates far from the planet or star are

$$\Omega - \kappa \approx +\frac{3}{2}J_2 \left(\frac{R_e}{r}\right)^2 \Omega,$$

$$\Omega - \Omega_z \approx -\frac{3}{2}J_2 \left(\frac{R_e}{r}\right)^2 \Omega.$$

[*Hint:* $P_n(x)$ satisfies Legendre's equation $(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0$. For positive integers m ,

$$P_{2m}(0) = \frac{(-1)^m(2m)!}{2^{2m}(m!)^2}.$$

For Saturn, $J_2 \approx 1.63 \times 10^{-2}$ and $J_4 \approx -9.4 \times 10^{-4}$.]

3. Motion in the Kerr metric

The exterior of a black hole is described by the Kerr metric, which contains two parameters: M , the mass of the black hole, and a , the dimensionless spin parameter, which satisfies $-1 < a < 1$. A relativistic treatment of test-particle orbits in the equatorial plane leads to the expressions

$$\Omega = \left(\frac{GM}{r^3}\right)^{1/2} \frac{1}{1 + ax^{-3/2}},$$

$$\kappa^2 = \Omega^2 (1 - 6x^{-1} + 8ax^{-3/2} - 3a^2x^{-2}),$$

$$\Omega_z^2 = \Omega^2 (1 - 4ax^{-3/2} + 3a^2x^{-2}),$$

where $x = r/(GM/c^2)$ is the orbital radius in gravitational units. (Here r is the Boyer–Lindquist radial coordinate, and the frequencies are those measured by an observer at infinity. The event horizon is located at $x = 1 + (1 - a^2)^{1/2}$. Since $\Omega > 0$, choosing $a > 0$ corresponds to a prograde orbit in the same direction as the spin of the black hole, while $a < 0$ corresponds to a retrograde orbit.)

- (i) Assuming the above expressions, deduce that circular orbits outside the event horizon, but sufficiently close to it, are unstable.

- (ii) Show that the leading approximations for the apsidal and nodal precession rates far from the black hole are

$$\Omega - \kappa \approx 3x^{-1}\Omega,$$

$$\Omega - \Omega_z \approx 2ax^{-3/2}\Omega.$$

4. Secular eccentricity evolution

The evolution equation for the complex eccentricity $E(\tau)$ reads (see lectures):

$$\frac{2}{\Omega}E_{,\tau} + iE = \frac{i}{r^3(\kappa^2 - \Omega^2)}(r^2\Phi')_{,r},$$

where Φ' is the eccentric component of the time-averaged perturbing potential and Ω is the circular velocity in the unperturbed potential. As a perturber, take a distant companion ($r_p \gg r$, subscripts p refer to the perturber) on an eccentric orbit so that

$$\Phi_p(r, \varphi, r_p, \varphi_p) = -\frac{GM_p}{\sqrt{r_p^2 + r^2 - 2rr_p \cos(\varphi - \varphi_p)}} = -GM_p \sum_{l=0}^{\infty} \frac{r^l}{r_p^{l+1}} P_l(\cos(\varphi - \varphi_p)),$$

where an expansion in Legendre polynomials is introduced in the last step. The $l = 1$ component can be discarded, since it cancels against the indirect term due to the acceleration of the coordinate frame centred on the primary star. We are interested in the time-averaged potential

$$\Phi_{\text{ave}} = \frac{1}{2\pi} \int_0^{2\pi} \Phi_p(r, \varphi, r_p, \varphi_p) d\varphi_p.$$

- (i) From Keplerian dynamics, we know that $a_p/r_p \approx 1 + e_p \cos(\varphi_p)$ (take $\varpi_p = 0$). Show that the $l = 0$ component does not contribute to the time-averaged potential (up to a constant).
- (ii) Calculate the time-averaged $l = 3$ component, keeping terms up to first order in eccentricity only. Relate this to Φ' .
- (iii) Calculate the time-averaged $l = 2$ component, keeping terms up to first order in eccentricity only. Relate this to $\kappa^2 - \Omega^2$.
- (iv) Calculate the forced eccentricity e_f and precession frequency $\kappa - \Omega$. Discuss the evolution of eccentricity qualitatively.

5. Linear diffusion equation

The surface density $\Sigma(r, t)$ of a Keplerian accretion disc evolves according to the diffusion equation

$$\frac{\partial \Sigma}{\partial t} = \frac{3}{r} \frac{\partial}{\partial r} \left[r^{1/2} \frac{\partial}{\partial r} (r^{1/2} \bar{\nu} \Sigma) \right], \quad (2)$$

where $\bar{\nu}$ is the mean effective kinematic viscosity.

Show that, for a particular choice of the function $\bar{\nu} = \bar{\nu}(r)$, equation (2) can be reduced to the classical diffusion equation

$$\frac{\partial g}{\partial t} = \frac{\partial^2 g}{\partial y^2}, \quad (3)$$

by a suitable transformation of variables.

Formulate the conservation laws of mass and angular momentum in terms of the variables g and y . What are the appropriate solutions of equation (3) representing the spreading of an initially very narrow ring subject to the boundary conditions that

- (i) mass can be accreted at $r = 0$, but no torque is exerted there?
- (ii) no mass is accreted at $r = 0$, but a torque is exerted there?

6. *Nonlinear diffusion equation*

- (i) Consider a disc with the viscosity law

$$\bar{\nu} = Ar^2\Sigma,$$

where A is a constant. Show that solutions of equation (2) of the form

$$\Sigma = \sigma(t) \left\{ \left[\frac{R(t)}{r} \right]^a - 1 \right\}, \quad r \leq R(t),$$

exist for only two non-zero values of the parameter a , namely $a = 1$ and $a = 5/4$. In each case, solve for $\sigma(t)$ and $R(t)$, assuming that $R(0) = 0$.

- (ii) For the solution with $a = 1$, show that the mean radial velocity in the disc is

$$\bar{u}_r = -\frac{(R - 5r)}{10t}.$$

Determine the trajectories of fluid elements moving with the mean radial velocity, and deduce that almost every fluid element is accreted in a finite time. For the solution with $a = 5/4$, show that \bar{u}_r is strictly positive.

- (iii) Investigate whether mass and angular momentum are globally conserved in either of the two solutions. Comment on the likely significance of these special solutions among solutions of the nonlinear diffusion equation as an initial-value problem with various initial and boundary conditions.

7. *Spreading of a narrow ring*

In the local approximation, with $r = r_0 + x$ and $|x| \ll r_0$, equation (2) reduces to

$$\frac{\partial \Sigma}{\partial t} = 3 \frac{\partial^2}{\partial x^2} (\bar{\nu} \Sigma).$$

If $\bar{\nu} \propto \Sigma^p$, with $p > 0$, show that algebraic solutions exist that are of the form

$$\Sigma = \sigma(t) \left\{ 1 - \left[\frac{x}{w(t)} \right]^a \right\}^b, \quad |x| \leq w(t),$$

for suitable choices of a and b . Solve for $\sigma(t)$ and $w(t)$. Comment on how mass and angular momentum are conserved in these solutions in the local approximation. Discuss the limit $p \rightarrow 0$.

8. *Vertical structure with radiation pressure*

The vertical structure of a thin, Keplerian accretion disc with constant (Thomson) opacity and a mixture of gas and radiation is governed by the equations

$$\begin{aligned} \frac{\partial p}{\partial z} &= -\rho \Omega^2 z, \\ \frac{\partial F}{\partial z} &= \mu \left(r \frac{d\Omega}{dr} \right)^2, \\ F &= -\frac{16\sigma T^3}{3\kappa\rho} \frac{\partial T}{\partial z}, \\ p &= \frac{k\rho T}{\mu_m m_p} + \frac{4\sigma T^4}{3c}. \end{aligned}$$

- (i) Show that there is only one possible value of the effective viscosity μ of an accretion disc in which the gas pressure is negligible compared to the radiation pressure. How does this value compare with the viscosity of water?
- (ii) According to one theory, whatever the ratio of gas and radiation pressures, the effective viscosity is always related to the gas pressure p_g by $\mu = \alpha p_g / \Omega$, where α is a constant. Show that the temperature then satisfies an equation of the form

$$\frac{1}{\rho} \frac{\partial}{\partial z} \left(\frac{1}{\rho} \frac{\partial T^4}{\partial z} \right) + A \Omega T = 0,$$

where A is a constant to be determined. By means of a suitable change of variables, find an explicit expression for the mean kinematic viscosity $\bar{\nu}(r, \Sigma)$ of the disc. You may assume that the ‘zero boundary conditions’ apply, and that there exists a unique non-trivial solution $t(\zeta)$ of the dimensionless boundary-value problem

$$\frac{d^2 t^4}{d\zeta^2} + t = 0, \quad t'(0) = t(1) = 0.$$

Your answer may involve an integral of $t(\zeta)$, which need not be evaluated.

9. Steady alpha disc

Write down the relation that must hold between $\bar{\nu}$ and Σ in a steady, Keplerian accretion disc with inner radius r_{in} (at which no torque is exerted) and accretion rate \dot{M} . Combine this with an order-of-magnitude treatment of the vertical structure of a gas pressure-dominated disc with Thomson opacity to deduce that

$$H \sim \alpha^{-1/10} \Omega^{-7/10} (f\dot{M})^{1/5} \left(\frac{\mu_{\text{m}} m_{\text{p}}}{k} \right)^{-2/5} \left(\frac{\sigma}{\kappa} \right)^{-1/10},$$

where $f = 1 - (r_{\text{in}}/r)^{1/2}$. (Notice how insensitive this result is to the poorly known value of α .) Hence show that the disc is very slightly flared, in the sense that H/r increases very slowly with increasing r .

Repeat the calculation for Kramers opacity, $\kappa = C_{\kappa} \rho T^{-7/2}$, to obtain

$$H \sim \alpha^{-1/10} \Omega^{-3/4} (f\dot{M})^{3/20} \left(\frac{\mu_{\text{m}} m_{\text{p}}}{k} \right)^{-3/8} \left(\frac{\sigma}{C_{\kappa}} \right)^{-1/20}.$$

(These results are approximately valid in the inner and outer parts, respectively, of accretion discs in interacting binary stars.)

Please send any comments and corrections to sjp88@damtp.cam.ac.uk