Part III: Galaxies
Lent Term 2010
Examples Sheet # 1: Dynamical Timescales

1. CROSSING TIME: Consider a spherical system with constant density ρ and radius R. (a) Show that the circular orbital period in such a system is independent of radius, and can be expressed as a function of the density alone. (b) Derive the equivalent expression for the crossing time (R/V) in terms of the total mass and radius of the system.

2. A more realistic approximation for a stellar system is an isothermal sphere, with constant orbital velocity v and radius R. Derive for this case an expression for the crossing time at outer radius R, in terms of R and the mass of the system, and compare to the result from the constant density case above. Explain the origin of any difference in timescales.

3. RELAXATION TIME: The crossing time represents a practical minimum time for altering the structure of a dynamical system. At the other end of the spectrum is the relaxation time, the time required for a system to reach dynamical equilibrium via 2-body interactions.

   (a) The relaxation time can be estimated by calculating the mean fractional change in the orbital kinetic energy of a star during a single crossing of the stellar system, and then scaling by the crossing time of the system. Begin by calculating the gravitational deflection of a star by passing by another star of equal mass, at impact parameter b. The force acting on the star perpendicular to its motion can be expressed as a function of time:

$$F_{\perp} = \frac{Gm^2}{b^2}\left[1 + \left(\frac{vt}{b}\right)^2\right]^{-3/2}$$

Show that over the entire encounter this introduces a transverse change in velocity:

$$\delta v_{\perp} = \frac{2Gm}{bv}$$

(b) Next consider a system comprised of N stars of equal mass m, and radius R. Derive an estimate for the total number of encounters that a star makes in one crossing of the cluster (with impact parameters between b and δb), and show that the total change in the square velocity is:

$$\delta v_{\perp}^2 = \left(\frac{2Gm}{bv}\right)^2 \frac{2N}{R^2} bdb$$

(c) To calculate the total change in $v_{\perp}^2$ we need to integrate over all impact parameters from $b_{\min} = Gm/v^2$ to R (collisions closer than $b_{\min}$ result in captures and are not relevant to this problem). Integrating gives:

$$\Delta V_{\perp}^2 = \int_{b_{\min}}^{R} \delta v_{\perp}^2 \approx 8N \left(\frac{Gm}{Rv}\right)^2 ln\left(\frac{R}{b_{\min}}\right)$$

Show the the fractional deviation per orbit crossing:

$$\frac{\Delta v_{\perp}^2}{v^2} = \frac{8ln\left(R/b_{\min}\right)}{N}$$

(d) Finally show that the relaxation time can be approximately expressed in terms of the crossing time as:

$$t_{relax} = \frac{N}{8lnN} t_{cross} = \frac{N}{8lnN} \frac{R}{v}$$

4. Use these results to estimate the values of the crossing and relaxation times for the following systems. For which cases are relaxation effects important?

(a)	the Galactic Center:	$N = 10^7$	$M = 10^7 M_\odot$	$R = 10\text{pc}$
(b)	elliptical galaxy M87:	$N = 10^{12}$	$M = 10^{12} M_\odot$	$R = 30\text{kpc}$
(c)	Coma cluster:	$N = 1000$	$M = 10^{14} M_\odot$	$R = 1\text{Mpc}$

5. DYNAMICAL FRICTION: As an extension of the analysis above, consider a large object of mass M orbiting through a sea of particles with average volume density ρ, at a velocity V. From the reference frame of the object, stars are streaming past at velocity v, and being gravitationally deflected (focussed) in a wake as they pass by the massive object. The result (back in the inertial rest frame) is a graviational drag $\delta V_{\|}$ on the massive object.

   To estimate the drag force, we use the formalism from Problem 3, and equate the momentum loss of the object to the net momentum loss of the stars being deflected. These losses are dominated by stars with impact parameters small enough that $\Delta v_{\perp} \geq V$.

   We denote this critical impact parameter as b_{crit}.

   Substituting from the second equation in Problem 3:

$$b_{crit} = \frac{2GM}{V^2}$$

   From this show that the net drag on the massive object:

$$\frac{dV}{dt} = -4\pi\rho \frac{G^2 M}{V^2}$$