Planetary System Dynamics Dr M. C. Wyatt
Mathematical Tripos Part III / Part III Astrophysics Lent Term 2014

Examples Sheet 4

1. *Secular evolution of orbital planes*
Consider a star of mass $M_\star$ with a planetary system comprised of two planets on circular orbits with masses and semimajor axes for the inner and outer planet of $[M_1, a_1]$ and $[M_2, a_2]$ respectively.

(a) The plane that is perpendicular to the total angular momentum vector of both planets' orbits is known as the invariable plane. Using the invariable plane as the reference plane show that the inclinations and longitudes of ascending node of the planets are related through

$$L_1 \sin I_1 = L_2 \sin I_2, \quad (1)$$
$$\Omega_2 - \Omega_1 = \pi, \quad (2)$$

where I_j, Ω_j and $L_j = M_j\sqrt{GM_\star a_j}$ are the inclination, longitude of ascending node and angular momentum of the j-th planet, respectively.

(b) Find expressions for I_1 and I_2 in terms of the mutual inclination $I \ll 1$ of the planets' orbits.

(c) To lowest order mutual secular perturbations cause the planetary system's complex inclination vector $\boldsymbol{y} = [y_1, y_2]$, where $y_j = I_j \exp i\Omega_j$, to evolve according to

$$\dot{\boldsymbol{y}} = i\boldsymbol{B}\boldsymbol{y}, \quad (3)$$

where $\boldsymbol{B}$ is a matrix with elements

$$B_{ji} = 0.25 n_j (M_i/M_\star)\alpha_{ji}\bar{\alpha}_{ji} b^1_{3/2}(\alpha_{ji}), \quad (4)$$
$$B_{jj} = -\sum_{i=1, i\neq j}^{N_{\rm pl}} B_{ji}, \quad (5)$$

n_j is the mean motion of the j-th planet, $\alpha_{ji} = \bar{\alpha}_{ji} = a_j/a_i$ for $a_i > a_j$ but $\alpha_{ji} = a_i/a_j$ and $\bar{\alpha}_{ji} = 1$ otherwise, and $b^1_{3/2}(\alpha_{ji})$ is a Laplace coefficient. Setting $\Omega_1 = 0$ at $t = 0$, solve equation (3) for the evolution of the planets' orbits to show that

$$\boldsymbol{y} = [I_1 \exp(-i(B_{12} + B_{21})t), -I_2 \exp(-i(B_{12} + B_{21})t)]. \quad (6)$$

(d) Draw a diagram showing the paths that these orbits take in the complex inclination plane and describe how mutual inclination varies with time.

2. *Test particle evolution / secular resonances*
The orbits of test particles in a planetary system evolve due to the secular perturbations of planets whose orbits also evolve due to their mutual perturbations. The secular evolution of a particle's complex inclination y is given by

$$\dot{y} = iBy + i\sum_{i=1}^{N_{\rm pl}} B_i y_i, \tag{7}$$

where B_i is given by B_{ji} from equation (4) dropping the subscript j, $B = -\sum_{i=1}^{N_{\rm pl}} B_i$, and y_i is the complex inclination of the i-th planet.

(a) Consider a test particle in the two planet system of question 1 for which the planets' orbits are described by equation (6). Solve for the evolution of the particle's complex inclination and sketch that evolution in the complex inclination plane.

(b) Show that for particles started in the invariable plane

$$y = I_f[\exp(-i(B_{12}+B_{21})t) - \exp(iBt)], \tag{8}$$

$$I_f = \frac{B_2 I_2 - B_1 I_1}{B_{12}+B_{21}-B_1-B_2}. \tag{9}$$

(c) The plane about which particle orbits precess is known as the forced inclination plane. What is the forced inclination plane for particles close to one of the planets?

(d) How many locations are there in the system where orbits started in the invariable plane remain there indefinitely? [You may find equation 1 useful.]

(e) Secular inclination resonances are special locations at which secular perturbations from the planets lead to the particle's inclination becoming infinite. Sketch $|B|$ as a function of a. By considering $|B|/(B_{12}+B_{21})$ prove that there are no secular inclination resonances between the two planets, and hence that there are just two secular inclination resonances in a two planet system.

(f) Optional: Assuming that the outermost secular resonance is located far from either planet, and noting that $b_{3/2}^1(\alpha) \approx (3/2)\alpha$ for $\alpha \ll 1$, show that the location of this resonance lies at a semimajor axis of

$$a_{\rm out} \approx a_1 \left[\frac{m_1\beta^2 + m_2}{m_1\beta^{9/2} + m_2\beta^4}\frac{3}{2b_{3/2}^1(\beta)}\right]^{2/7}, \tag{10}$$

where $\beta = a_1/a_2$.

3. <u>*Resonant geometry / asymmetric libration*</u>
The Kuiper belt object $2003LG_7$ was found to have orbital parameters of $a = 63.0$AU, $e = 0.485$, $\varpi = 220.8°$, with a mean anomaly at the epoch of discovery of $M = 7.6°$. Assume the orbit of Neptune, the mass of which is $5.1 \times 10^{-5} M_\odot$, is fixed at $a_N = 30.1$AU, $e_N = 0$, and that it was at a longitude of $\lambda_N = 304.9°$ at the epoch of discovery.

(a) Considering all motion to be in the same plane, and ignoring any dynamical evolution of the KBO's orbit, find and comment on the minimum possible closest approach distance between the two objects in units of Neptune's Hill radius, which is defined as $r_\mathrm{H} = a_\mathrm{pl}(\frac{M_\mathrm{pl}}{3M_\star})^{1/3}$.

(b) Which resonant argument ϕ is likely to be dominating the orbital motion and what is its current value?

(c) Assuming the KBO is in this resonance sketch the path of its orbit in the frame rotating with Neptune, noting the angle ϕ on this plot.

(d) Using Appendix B of Murray & Dermott (1999) identify the relevant terms in the disturbing function up to and including terms of order $O(e^4)$, then use Lagrange's planetary equation for the evolution of the semimajor axis

$$da/dt = \frac{2}{na}\partial\mathcal{R}/\partial\lambda \tag{11}$$

to show that the resonant argument is expected to librate around ϕ_m where

$$\cos\phi_\mathrm{m} = A + Be^{-2}, \tag{12}$$

and A and B are functions of a/a_N that should be determined.

(e) Given that $A = -0.0137$, $B = -0.0392$ and $f_{94} > 0$, comment on whether the KBO's semimajor axis is consistent with libration about ϕ_m.

4. *Resonance width*
Consider particles orbiting a planet of mass $M_{\rm pl}$ near a first order $j-1:j$ resonance with an exterior satellite of mass $M_{\rm sat}$ that is orbiting at a semimajor axis $a_{\rm sat}$. For a particle with an orbit defined by semimajor axis a, eccentricity e, longitude of pericentre ϖ and mean longitude λ, the perturbations it experiences from the satellite are dominated by the resonant term in the disturbing function

$$\mathcal{R} = (GM_{\rm sat}/a_{\rm sat}) f(a/a_{\rm sat}) e \cos\phi, \tag{13}$$

where $\phi = j\lambda_{\rm sat} - (j-1)\lambda - \varpi$ is the resonant angle $\lambda_{\rm sat}$ is the mean longitude of the satellite and f is some function of $a/a_{\rm sat}$.

(a) Use the lowest order form of Lagrange's planetary equations to show that the evolution of the particle's orbital elements can be written

$$\dot{a} = -2Cae(1-j)\sin\phi, \tag{14}$$
$$\dot{e} = -C\sin\phi, \tag{15}$$
$$\dot{\varpi} = Ce^{-1}\cos\phi, \tag{16}$$
$$\dot{\epsilon}^* = 0.5Ce\cos\phi, \tag{17}$$

where ϵ^* is the mean longitude at epoch, $C = (M_{\rm sat}/M_{\rm pl})(a/a_{\rm sat})f(a/a_{\rm sat})n$ and n is the particle's mean motion.

(b) Ignoring changes in mean longitude at epoch show that the rate of change of resonant angle is given by

$$\dot{\phi} \approx 1.5jxn_{\rm sat} - Ce^{-1}\cos\phi, \tag{18}$$

where $n_{\rm sat}$ is the satellite's mean motion, $x = (a - a_{\rm r})/a_{\rm r}$ and $a_{\rm r}$ is the nominal location of the resonance.

(c) Show that the acceleration of ϕ can be written in the form

$$\ddot{\phi} = (A + B\cos\phi)\sin\phi, \tag{19}$$

where A and B are functions of the orbital parameters of the particle and satellite that should be determined.

(d) Show that there are two fixed points at which $\dot{\phi} = \ddot{\phi} = \dot{a} = \dot{e} = 0$ that are located at

$$x = \pm\frac{2C}{3n_{\rm sat}ej}. \tag{20}$$

(e) Sketch the orbits of particles at the fixed points in the frame rotating with the satellite.

(f) Find the critical separation from the resonance $x_{\rm crit}$ within which the orbits at the two fixed points overlap and comment on the implications for particles near first order mean motion resonances.

(g) Assess the stability of the fixed points.

5. *Evolution near resonance / Pulsar planets*

The first known extrasolar planetary system was discovered by timing measurements of the millisecond pulsar PSR1257+12. The system consists of three planets, though one of these is so low in mass that it can be ignored for the purposes of this question. Assuming a mass of $1.4M_\odot$ for the pulsar the timing observations were used to derive the following parameters for the two most massive planets b and c

	$M_{\rm pl}\sin I$	$a_{\rm pl}$	$e_{\rm pl}$	$\varpi_{\rm pl}$
b	$3.4M_\oplus$	0.3595 AU	0.022	252°
c	$2.8M_\oplus$	0.4660 AU	0.020	107°

where $M_\oplus$ is the mass of the Earth.

(a) How do the planets' masses and semimajor axes scale with the true stellar mass $M_\star$? [You may find it useful to refer back to question 2 of examples sheet 1.]

(b) The two planets b and c are near (but not in) 3:2 resonance. Describe the evolution of the resonant arguments $\phi_1 = 3\lambda_c - 2\lambda_b - \varpi_b$ and $\phi_2 = 3\lambda_c - 2\lambda_b - \varpi_c$.

(c) Assuming the orbits to be coplanar use Appendix B of Murray & Dermott (1999) to identify the relevant terms in the disturbing function keeping terms up to second order in eccentricities, and then Lagrange's planetary equations to get expressions for $\dot{a}_b$, $\dot{a}_c$, $\dot{e}_b$, $\dot{e}_c$, $\dot{\varpi}_b$ and $\dot{\varpi}_c$.

(d) Hence assuming the orbital elements to be slowly varying show that their evolution can be written in the form

$$x(t) = A + Bt + C\sin(Dt + E), \qquad (21)$$

where x is one of a_b, a_c, e_b, e_c, ϖ_b, ϖ_c.

(e) Equation 21 can be used to provide a prediction for changes in the timing measurements and so confirm whether this signal does indeed arise from a planetary system. What is the period $2\pi/D$ of the expected oscillations and how is it affected by uncertainties in the stellar mass or inclination of the planets' orbital plane to the line-of-sight?

(f) Can measurements of the amplitudes of the oscillations C and/or of the linear drift B be used to constrain both the inclination of the planets' orbital plane to the line-of-sight and the planet masses?