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Wave Propagation and Scattering - 12 lectures of 24
Part III

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May 11, 2012

This part of course deals with propagation and scattering of acoustic waves by inhomogeneous, possibly random, media and by rough surfaces. The direct problem of calculating the scattered field, given an incident field and a scatterer (which could be a surface or an extended medium) will be considered first. The last two chapters are concerned with the inverse problem of calculating properties of the scatterer from measurements of the scattered field. The linear approximation to the wave equation for acoustic waves will be used throughout.

This is a revised draft of the material covered in this course. I am very grateful to Andrew McRae, who spotted several misprints. As a result, this should be a much better version, though by no means guaranteed free of misprints. I should very much appreciate being told of any further corrections or possible improvements. Comments, please, to O.Rath-Spivack@damtp.cam.ac.uk.

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Part I
Wave Equations

We shall use throughout the linearised wave equation, and restrict ourselves to time-harmonic waves only, i.e. those with time dependence $\exp(-i\omega t)$, so that our starting point will generally be the **Helmholtz equation**

$$\left(\nabla^2 + k^2\right) \psi = 0 \ , \tag{0.1}$$

and all analysis will be restricted to the frequency domain. Since this course is concerned with acoustic waves, the function ψ in (0.1) can represent alternatively pressure, density, or velocity potential. Just as an aside, it's worth mentioning that of course the wave equation can describe all sort of waves, e.g. waves on a vibrating string, in which case ψ is displacement, or electromagnetic waves, in which case ψ can be the (vector) electric field, or magnetic field, or appropriate scalar and vector potentials.
In the context of time-harmonic acoustic waves, then, we shall derive a few important equations that will be useful for solving different problems of wave propagation and scattering in inhomogeneous and random media.

1 Introduction

We shall first recall a few main results and establish notation.

Calculations will be done for an acoustic field ψ, which shall generally be taken as the complex velocity potential. We should therefore be careful to take

$$\begin{aligned} p &= \mathrm{Re}\left[i\omega\rho\psi\exp(-i\omega t)\right] \\ \mathbf{v} &= \mathrm{Re}\left[\nabla\psi\exp(-i\omega t)\right] \end{aligned} \tag{1.1}$$

when dealing with real physical quantities p and $\mathbf{v}$, pressure and velocity.
Working in the frequency domain, we shall generally drop the time-dependent part of the wave. Results will nevertheless be readily extended to non-monochromatic waves, since any acoustic field $\psi(\mathbf{x}, t)$ can be written as a superposition of time-harmonic waves. This can be done using a Fourier transform $\hat{\psi}(\mathbf{x}, \omega)$ (as long as $\mid \psi(\mathbf{x}, t) \mid$ and $\mid \hat{\psi}(\mathbf{x}, \omega) \mid \in L^2$):

$$\psi(\mathbf{x}, t) = \int_{-\infty}^{\infty} \hat{\psi}(\mathbf{x}, \omega) \exp(-i\omega t) d\omega \tag{1.2}$$

where

$$\hat{\psi}(\mathbf{x},\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(\mathbf{x},t)\exp(i\omega t)dt \tag{1.3}$$

Since we are using the Helmholtz equation, which is linear, each Fourier component also obeys the Helmholtz equation, and the total field can be reconstructed after solving the scattering problem for whatever boundary conditions on any finite surfaces are appropriate.

In the time domain, a causality condition is introduced to reflect the physical fact that an acoustic signal cannot have an effect before it's switched on. In the frequency domain, the causality condition cannot be expressed as a condition in time, and takes the form of the **Sommerfeld radiation condition**. Causality then is expressed by the integrability condition implicit in assuming that a Fourier representation of the wave exists, and becomes a condition in space (boundary condition at infinity):

$$\psi(\mathbf{x}) = \mathcal{O}(\mid \mathbf{x} \mid^{-1/2}) \tag{1.4}$$

or, more usually:

$$|\mathbf{x}| \left(\frac{\partial \psi(\mathbf{x})}{\partial |\mathbf{x}|} - ik\psi(\mathbf{x}) \right) \to 0 \tag{1.5}$$

uniformly as $|\mathbf{x}| \to \infty$. This expresses the requirement that the field should contain no incoming waves as $\mid \mathbf{x} \mid \to \infty$. In general, integrability, hence causality, will also result in restrictions imposed on the contour chosen for the integration in the complex plane.

Here we have assumed 3D space. In general, for n-dimensional space, the factor $|\mathbf{x}|$ in (1.5) needs to be replaced by $|\mathbf{x}|^{(n-1)/2}$.

We shall consider problems where there will be one or more sources of sound, and the space where the problem needs to be solved will include one or more surfaces. Consequently, the differential equation to be solved will be an inhomogeneous version of (0.1), and the solutions will be subject to other boundary conditions in addition to (1.5). In general the problem in question will then be defined by a differential equation

$$\nabla^2\psi(\mathbf{x},t) + k^2\psi(\mathbf{x},t) = f(\mathbf{x},t) \ , \tag{1.6}$$

together with boundary conditions on one or more surfaces and the Sommerfeld conditions. We shall often use an appropriate Green's function when deriving solutions for scattering problems.

Boundary conditions (b.c.), expressed by constraints on the values taken by the field and its normal derivative at a surface, will reflect the properties of the surface.

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For perfectly reflective surfaces we have two possible cases:
Neumann condition, when the normal derivative of the potential field is given at the boundary, i.e., if $\mathbf{n}$ is the unit normal pointing outward from the surface:

$$\frac{\partial \psi(\mathbf{r})}{\partial n} = 0, \ \mathbf{r} \text{ on } S. \tag{1.7}$$

This corresponds to an acoustically hard surface.
Dirichlet condition, when the value of the potential field is given at the boundary:

$$\psi(\mathbf{r}) = 0, \ \mathbf{r} \text{ on } S. \tag{1.8}$$

which corresponds to a pressure-release or acoustically soft surface.

In most real cases the surface is not perfectly reflecting and both the potential and its normal derivative are different from zero at the boundary. It is then convenient to express the boundary condition as an approximate equation relating these two quantities. This is called ***Cauchy condition*** (or ***Robin***, or ***impedance boundary condition***) and is usually expressed by

$$\frac{\partial \psi(\mathbf{r})}{\partial n} = iZ(\mathbf{r}, \omega, \theta, ...)\psi(\mathbf{r}) \ \mathbf{r} \text{ on } S. \tag{1.9}$$

The impedance of the surface, Z, usually varies with the incoming field at each point. In general, Z is also a function of frequency and angle of incidence.
Often a surface will be an interface between two different media, with the boundary conditions reflecting the continuity of actual physical quantities at the interface, and the characteristic properties of the two media.
Let's call the two media medium 1 and medium 2, with densities ρ_1 and ρ_2. The b.c. must reflect continuity of pressure, which means that there cannot be a net force at the interface, and continuity of the normal component of the velocity, which means that the two media are in contact at the interface (no gaps). These are normally expressed in terms of the velocity potential by the 'jump conditions':

$$\begin{aligned} \rho_1\psi_1 &= \rho_2\psi_2 \\ \frac{\partial \psi_1}{\partial n} &= \frac{\partial \psi_2}{\partial n} \end{aligned} \tag{1.10}$$

where the subscripts 1 and 2 refer to the two media, and we take $\mathbf{n}$ as the normal directed into medium 1.

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The different types of scattering problems involving either impenetrable or penetrable scatterers (so perfectly reflecting or not perfectly reflecting surfaces) can be classified as follows below. Here we shall denote by V_0 a bounded domain with boundary S and exterior V.

Exterior Dirichlet boundary value problem
Find a function ψ which satisfies the Helmholtz equation in V, the Sommerfeld radiation condition at infinity, and the boundary condition

$$\psi = f \text{ on } S \ , \tag{1.11}$$

where f is a given continuous function defined on S, and in particular we shall also require f to have continuous first derivative (because we shall need its normal derivative).
Interior Dirichlet boundary value problem
Similar to the problem above, but seeking a solution in V_0, and without imposing the Sommerfeld radiation condition.

The relation between ψ at the boundary and its normal derivative $\frac{\partial\psi}{\partial n}$ can be written as:

$$\frac{\partial\psi}{\partial n} = Af \ , \tag{1.12}$$

where the operator A is called the ***Dirichlet to Neumann map***, since it takes Dirichlet boundary data to the normal derivative of the field at the boundary, i.e. Neumann data.

Exterior Neumann boundary value problem
Find a function ψ which satisfies the Helmholtz equation in V, the Sommerfeld radiation condition at infinity, and the boundary condition

$$\frac{\partial\psi}{\partial n} = g \text{ on } S \ , \tag{1.13}$$

where g is a given continuous function defined on S, and again we shall also require g to have continuous first derivative.
Interior Neumann boundary value problem
Similar to the problem above, but seeking a solution in V_0, and without imposing the Sommerfeld radiation condition.

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2 The Kirchoff-Helmholtz equation

By using the Green's function it is possible to derive an integral form of the Helmholtz equation which facilitates calculations of sound propagation and scattering and allows sources and boundary conditions to be treated in a simple and convenient way.
In order to derive this integral equation, we shall first recall the following vector identities. Given any two functions f and g, we have:

$$\nabla \cdot (f\nabla g) = f\nabla^2 g + (\nabla f)\cdot(\nabla g) \ . \tag{V1}$$

If $f\nabla g$ is a vector field continuously differentiable to first order, which we shall denote by $\mathbf{F} = f\nabla g$, then we can apply to it the following theorem, which transforms a volume integral into a surface integral:
Gauss theorem *If V is a subset of $\mathbb{R}^n$, compact and with piecewise smooth boundary S, and $\mathbf{F}$ is a continuously differentiable vector field defined on V, then*

$$\int_V \nabla\cdot\mathbf{F}\,dV = \int_S \mathbf{F}\cdot\mathbf{n}\,dS \ , \tag{V2}$$

where $\mathbf{n}$ is the outward-pointing unit normal to the boundary S.
In $\mathbb{R}^3$, for an $\mathbf{F}_1 = f\nabla g$ and an $\mathbf{F}_2 = g\nabla f$, we have, using V2 and V1:

$$\int_V \left[f\nabla^2 g + (\nabla f)\cdot(\nabla g)\right]\,dV = \int_{\partial V} f\nabla g\cdot\mathbf{n}\,dS \ , \tag{2.1}$$

$$\int_V \left[g\nabla^2 f + (\nabla g)\cdot(\nabla f)\right]\,dV = \int_{\partial V} g\nabla f\cdot\mathbf{n}\,dS \ , \tag{2.2}$$

and subtracting (2.2) from (2.1) we obtain:

$$\int_V \left(f\nabla^2 g - g\nabla^2 f\right)\,dV = \int_{\partial V} (f\nabla g - g\nabla f)\cdot\mathbf{n}\,dS \ . \tag{2.3}$$

This result is variously referred to as ***Green's theorem*** or ***Green's second identity***, and can be used to solve a general scattering problem involving one or more sources, and write the solution at any point in V in terms of the (unknown) field and its normal derivative along the boundary. The integral equations obtained can in principle be solved to find these unknown surface field values. This approach applies whether the problem involves an interface with a vacuum or with a second medium.

Let's consider then the field ψ generated by a source $Q(\mathbf{r})$:

$$\nabla^2\psi(\mathbf{r}) + k^2\psi(\mathbf{r}) = -Q(\mathbf{r}) \ . \tag{2.4}$$

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We shall apply Green's theorem (2.3) to a volume V contained between two smooth closed surfaces S and S_∞ and containing a source $Q(\mathbf{r})$. We further denote the integration variable as $\mathbf{r}'$, and let $\partial/\partial\mathbf{n}'$ be the normal derivative pointing *inward* into V, and identify

$$\begin{aligned} f(\mathbf{r}') &= \psi(\mathbf{r}') \\ g(\mathbf{r}') &= G(\mathbf{r},\mathbf{r}')\ , \end{aligned} \tag{2.5}$$

where ψ is the solution to (2.4), and G is the free space Green's function, i.e. G satisfies

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + k^2 G(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}') \tag{2.6}$$

For reasons that will become clear very soon, we shall also introduce the surface S_1 of a small ball of radius ϵ centred around a point $\mathbf{r}$ in V, which shall be our observation point.

In V, $G(\mathbf{r},\mathbf{r}')$ satisfies the homogenous wave equation

$$\nabla^2 G(\mathbf{r},\mathbf{r}') + k^2 G(\mathbf{r},\mathbf{r}') = 0$$

because we have excluded the observation point.
In V we can now write

$$\begin{aligned} \psi(\mathbf{r}')\nabla'^2 G(\mathbf{r},\mathbf{r}') &- G(\mathbf{r},\mathbf{r}')\nabla'^2\psi(\mathbf{r}') = \\ &= \psi(\mathbf{r}')(\nabla'^2 + k^2)G(\mathbf{r},\mathbf{r}') - G(\mathbf{r},\mathbf{r}')(\nabla'^2 + k^2)\psi(\mathbf{r}') = G(\mathbf{r},\mathbf{r}')Q(\mathbf{r}')\ , \end{aligned} \tag{2.7}$$

where ∇' denotes differentiation w.r.t. $\mathbf{r}'$.
Integrating (2.7) over V and using Green's theorem we have:

$$\psi_i = -\int_{\partial V}\left(\psi(\mathbf{r}')\frac{\partial G(\mathbf{r},\mathbf{r}')}{\partial n'} - \frac{\partial\psi(\mathbf{r}')}{\partial n'}G(\mathbf{r},\mathbf{r}')\right) ds'\ , \tag{2.8}$$

where we have used $\frac{\partial}{\partial n'} = \mathbf{n}'\cdot\nabla' = -\frac{\partial}{\partial n}$, and the results that G satisfies the homogeneous Helmholtz equation in V, and that

$$\psi_i(\mathbf{r}) = \int_V Q(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}'. \tag{2.9}$$

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is the incident field ψ_i.
The boundary ∂V comprises the surfaces S_∞, S_1 and S, so that the surface integral in (2.8) is

$$\int_{\partial V} = \int_{S_\infty} + \int_{S_1} + \int_S \tag{2.10}$$

We shall now let the outer surface extend to infinity, and the surface integral will become an integral over S_1 and S only, since

$$\int_{S_\infty} (\ldots) \to 0 \text{ as } S_\infty \to \infty \ ,$$

because of the Sommerfeld boundary condition at infinity.
We now want to let $\epsilon \to 0$, since we need to include the observation point. In 3D, the free space Green's function $G(\mathbf{r}, \mathbf{r}')$ is:

$$G(\mathbf{r}, \mathbf{r}') = \frac{e^{ik\epsilon}}{4\pi\epsilon} \ , \tag{2.11}$$

where $\epsilon =\mid \mathbf{r} - \mathbf{r}' \mid$, which has a singularity at $\mathbf{r} = \mathbf{r}'$.
In the limit $\epsilon \to 0$, we then have:

$$\begin{aligned} &\lim_{\epsilon \to 0} \left[\int_{S_1} \psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} ds' - \int_{S_1} \frac{\partial \psi(\mathbf{r}')}{\partial n'} G(\mathbf{r}, \mathbf{r}') \right] ds' \qquad (2.12) \\ = \ &\lim_{\epsilon \to 0} \left[\psi(\mathbf{r}) \frac{\partial}{\partial \epsilon} (\frac{e^{ik\epsilon}}{4\pi\epsilon}) 4\pi\epsilon^2 \right] = -\psi(\mathbf{r}) \ . \end{aligned}$$

From (2.13) and (2.8) we have, for $\mathbf{r}$ in V:

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_S \left[\psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial \psi}{\partial n'}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \right] ds' \ . \tag{2.13}$$

This is the **Kirchoff-Helmholtz equation**, an integral (implicit) form of the Helmholtz equation, which is of great practical use in calculating the field induced by sources scattered by finite boundaries. (2.13) is valid for $\mathbf{r}$ in V, so gives us an expression for the total field *outside* the scatterer enclosed by S as a sum of the incident field ψ_i and a scattered field.

Case when the observation point $\mathbf{r}$ is *on* the boundary S.
We want to move $\mathbf{r}$ to $\mathbf{r}'$, so let's move our infinitesimal sphere with surface S_1 surrounding $\mathbf{r}$ onto the surface S, by making an infinitesimal hemispherical indentation S_2 in S

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Since the surface S_2 is just half of S_1, from (2.13) we have

$$\lim_{\epsilon \to 0} \left[\int_{S_1} \psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} ds' - \int_{S_1} \frac{\partial \psi(\mathbf{r}')}{\partial n'} G(\mathbf{r}, \mathbf{r}') \right] = \frac{1}{2} \psi(\mathbf{r}) \ ,$$

where the sign comes from having normal derivatives in opposite directions on S_1 and S_2. For $\mathbf{r}$ on the boundary, then, we have:

$$\frac{1}{2} \psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_S \left[\psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial \psi}{\partial n'}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \right] ds' \ , \tag{2.14}$$

where the surface integral should be interpreted as the 'principal part' of the integral, which means we need to take a limit at the singularity.

Case when the observation point $\mathbf{r}$ is *inside* the boundary S.
If $\mathbf{r}$ is inside S, then the whole of the infinitesimal surface S_1 is also inside S, and the surface S_1 is not part of the boundary ∂V. Therefore we have:

$$0 = \psi_i(\mathbf{r}) + \int_S \left[\psi(\mathbf{r}') \frac{\partial G(\mathbf{r}, \mathbf{r}')}{\partial n'} - \frac{\partial \psi}{\partial n'}(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') \right] ds' \ . \tag{2.15}$$

This latest result is also known as the **extinction theorem**, since it says that inside S the sum of the incident and scattered field is zero, i.e. the incident field inside the surface is 'extinguished' by the surface contribution.

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3 Paraxial Approximation and Parabolic Equation

Consider first a *scalar plane wave ψ in* **free space** (where we again assume and suppress a time-harmonic variation $e^{-i\omega t}$), with wavenumber k in a two-dimensional medium (x, z). Here x is horizontal and z is vertical. So ψ obeys the Helmholtz wave equation $(\nabla^2 + k^2)\,\psi = 0$. Suppose that ψ is propagating at a small angle α to the horizontal, say

$$\psi(x, z) = e^{ik(x\cos\alpha + z\sin\alpha)}\ . \tag{3.1}$$

Since $\sin\alpha$ is small we can approximate

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} \cong 1 - \sin^2\alpha/2.$$

Now the fastest variation of ψ is close to the x direction, so define the 'slowly-varying' part E of ψ by

$$E = \psi e^{-ikx}$$

so that

$$E \cong e^{ik(-x\sin^2\alpha/2 + z\sin\alpha)}. \tag{3.2}$$

(E is also referred to as the reduced wave.) It then follows that

$$\frac{\partial E}{\partial x} = \frac{i}{2k}\frac{\partial^2 E}{\partial z^2}. \tag{3.3}$$

This is one form of the ***parabolic wave equation in free space***, and we have derived it here for the special case where ψ is a plane wave, but it holds for any superposition of plane waves travelling at small angles to the horizontal.

It is straightforward to write the exact solution of (3.3) in terms of an initial value by using Fourier transforms.

Let E be a field obeying (3.3). Define the Fourier transform of E with respect to z,

$$\hat{E}(x, \nu) = \int_{-\infty}^{\infty} E(x, z)e^{i\nu z}\, dz. \tag{3.4}$$

Note that this is equivalent to looking at the inverse Fourier transform $E(x, z)$ as a superposition of plane waves with spectral component $\hat{E}(x, \nu)$ at a vertical plane x.

Taking the z-transform of (3.3) gives an equation for $\hat{E}$,

$$\frac{\partial \hat{E}}{\partial x} \;=\; -\frac{i\nu^2}{2k}\,\hat{E}. \tag{3.5}$$

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This has solution (in terms of E at the vertical plane $x = 0$)

$$\hat{E}(x,\nu) \;=\; e^{-i\nu^2 x/2k}\;\hat{E}(0,\nu). \tag{3.6}$$

Equation (3.3) can also be derived by substituting the form $E = \psi e^{ikx}$ into the Helmholtz wave equation for ψ, and neglecting terms of the form $\partial^2 E/\partial x^2$.

We shall now consider the more general case of a harmonic plane wave source **in a refractive medium**, again in 2D. The Helmholtz equation is therefore

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} + k_0^2 n^2 \psi = 0 \;, \tag{3.7}$$

where $k_0 = \omega/c_0$ is a reference wave number, and $n(x,z) = c_0/c(x,z)$ is the index of refraction of the medium.
Considering first the case when the field is a plane wave, we shall define the reduced wave as

$$E = \psi e^{-ik_0 x} \;.$$

This satisfies

$$\frac{\partial^2 E}{\partial x^2} + 2ik_0\frac{\partial E}{\partial x} + \frac{\partial^2 E}{\partial z^2} + k_0^2(n^2-1)E = 0. \tag{3.8}$$

This equation can be factorised formally in terms of the operators

$$A = \frac{\partial}{\partial x}$$

and

$$B = \sqrt{\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + n^2}$$

as

$$(A + ik_0(1-B))(A + ik_0(1+B))E - ik_0[A,B]E = 0 \;. \tag{3.9}$$

Under the assumption that A and B nearly commute: $[A,B] \approx 0$, which is equivalent to assuming that the refractive index is very slowly varying in the x-direction, $n(x,z) \approx n(z)$, and selecting only the term in (3.9) corresponding to outgoing waves, we obtain

$$AE = -ik_0(1-B)E \;, \tag{3.10}$$

which has formal solution

$$E(x,z) \;=\; e^{-ik_0(1-B)x}\;E(0,z). \tag{3.11}$$

This can only be used in practice by introducing a series expansion for the operator B. To do this we shall make a further approximation, consistent with the original approximation of propagation at small angles, and assume small variation of the refractive index n: $n \approx 1$. Therefore we write B as

$$B = \sqrt{1+b} \ , \tag{3.12}$$

where

$$b = \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + n^2 - 1 \ , \tag{3.13}$$

then, if b is small, we can Taylor expand B and keep the first 2 terms to give the approximation

$$B \simeq 1 + \frac{b}{2} = 1 + \frac{1}{2k_0^2}\frac{\partial^2}{\partial z^2} + \frac{n^2-1}{2} \ . \tag{3.14}$$

Substituting this expression into (3.10) we obtain the desired ***parabolic equation in a refractive medium***

$$\frac{\partial E}{\partial x} = \frac{i}{2k_0}\frac{\partial^2 E}{\partial z^2} + \frac{ik_0}{2}(n^2-1)E \ . \tag{3.15}$$

It is seen here that the effect of the medium is contained in the second term on the right hand side. The first term on the right is therefore often thought of as the *diffraction term*, and the second as the *scattering term*.
Other forms of the parabolic wave equation can be obtained by using different approximations for the square root operator.

The parabolic form of the wave equation just derived can easily be solved with efficient numerical techniques, and is very useful in a variety of real problems. It is widely used, for example in tropospheric radiowave propagation, since $n \approx 1$ in air, and the angles of interest are usually less than a few degrees.
Nevertheless, the error involved is large for large angles (e.g. error $> 10^{-2}$ for $\alpha = 20°$), since the approximation used above in obtaining the parabolic equation leads to an error proportional to $\sin^4 \alpha$. One could in principle obtain better approximations by using higher order terms in the Taylor expansion for the operator B, but it turns out that this leads to instabilities in the numerical solvers.

Suitable more accurate expansions for large angles are obtained in terms of Padé approximants, and are referred to as ***wide-angle methods***.
Approximating the square root operator with a Padé approximant of the form

$$B = \sqrt{1+b} = \frac{1+pb}{1+qb} \tag{3.16}$$

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leads to and error proportional to $\sin^6(\alpha)$.
Approximating the exponential operator which appears in the formal solution directly with a Padé approximant of the form

$$e^{ikx\sqrt{1+b}} \sim 1 + \sum_{l=1}^{N} \frac{p_l b}{1 + q_l b} \tag{3.17}$$

leads to a stable numerical scheme that allows to increase the angular range of validity according to the number N of terms in (3.17).

Summary

Advantages:
• The parabolic wave equation replaces a second order equation with a first order one

• The parabolic wave equation replaces a boundary-value problem with an initial-value problem

Assumptions:
• The energy propagates at small angles to a preferred directions (the paraxial direction).

• $\mid \frac{\partial^2 \psi}{\partial x^2} \mid \ll k \mid \frac{\partial \psi}{\partial x} \mid$.

• The operators $A = \frac{\partial}{\partial x} + ik_0$ and $B = \sqrt{\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + n^2}$ nearly commute equivalently
• The variation of the refractive index n remains slow on the scale of a wavelength).

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4 Born Approximation

The Born approximation is based on expressing the total wave field ψ, which is in general the solution of a scattering problem in a volume with sources and surfaces, as the sum of the incident field plus a 'small' perturbation:

$$\psi = \psi_i + \psi_s \ , \tag{4.1}$$

The actual solution in this approximation will take various forms, depending on how the perturbation is expressed.
We can immediately see how the Born approximation can be applied to the integral form of the wave equation (2.13), to obtain a first Born approximation

$$\psi^{(1)}(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_{S_0} \left[\psi_i(\mathbf{r}_0) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - \frac{\partial \psi_i}{\partial n}(\mathbf{r}_0) G(\mathbf{r}, \mathbf{r}_0) \right] d\mathbf{r}_0 \ , \tag{4.2}$$

and higher terms can be obtaind by iteration.
The Born approximation will only be valid when $\psi_s \ll \psi_i$, which intuitively must apply to some kind of 'weak scattering'. In order to understand better what this means in practice, to relate it to the physical features of a scattering problem, and find boundaries for its range of validity, we shall derive it here for some particular cases.
We shall consider the case where the scattered field is the result of a varying refractive index $n(\mathbf{r})$. The total field satisfies

$$\nabla^2 \psi + k^2(\mathbf{r}) \psi = 0 \ . \tag{4.3}$$

We can then write

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0(1 + n_\delta(\mathbf{r})) \ , \tag{4.4}$$

where it is assumed $n_\delta(\mathbf{r}) \ll 1$. Substituting $k_0 n(\mathbf{r})$ into (4.3) we get:

$$\nabla^2 \psi + k_0^2 \psi = -k_0^2 (n^2(\mathbf{r}) - 1) \psi \equiv -V(\mathbf{r}) \psi \ . \tag{4.5}$$

Using (4.1), and the fact that the incident field satisfies

$$\nabla^2 \psi_i + k_0^2(\mathbf{r}) \psi_i = 0 \ , \tag{4.6}$$

we can write the wave equation for the scattered wave

$$\nabla^2 \psi_s + k^2(\mathbf{r}) \psi_s = -V(\mathbf{r}) \psi \ . \tag{4.7}$$

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We can then solve for ψ_s using the free space Green's function, with $-V(\mathbf{r})\psi$ as the source term

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r}-\mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ . \tag{4.8}$$

But $\psi_s = \psi - \psi_i$, so

$$\psi = \psi_i(\mathbf{r}) + \int G(\mathbf{r}-\mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ . \tag{4.9}$$

We can write the above implicit integral equation as an infinite series of explicit integral equations by forming successive approximations starting from the unperturbed incident field ψ_i:

$$\begin{aligned}
\psi^{(0)} &= \psi_i \\
\psi^{(1)} &= \psi_i(\mathbf{r}) + \int G(\mathbf{r}-\mathbf{r}')[V(\mathbf{r}')\psi^{(0)}(\mathbf{r}')]d\mathbf{r}' \\
\psi^{(2)} &= \psi_i(\mathbf{r}) + \int G(\mathbf{r}-\mathbf{r}')[V(\mathbf{r}')\psi^{(1)}(\mathbf{r}')]d\mathbf{r}' \\
\psi^{(3)} &= \ldots
\end{aligned}$$

The first iteration in this series, $\psi^{(1)}$, is know as the *first-order Born approximation*, usually referred to just as Born approximation.
This can also be put in a more compact form by writing the integration with Green's function as an operator:

$$\int G(\mathbf{r}-\mathbf{r}')[f(\mathbf{r}')]d\mathbf{r}' \equiv \hat{G}f$$

so (4.9) becomes $\psi = \psi_0 - \hat{G}V\psi$, and the series becomes

$$\begin{aligned}
\psi^{(0)} &= \psi_i \\
\psi^{(1)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} \\
\psi^{(2)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} + \hat{G}V\hat{G}V\psi^{(0)} \\
&\ldots \\
\psi^{(n)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} + \cdots + (\hat{G}V)^n\psi^{(0)}
\end{aligned}$$

This form of the Born series helps visualising the structure of the n-th order approximation, and is the one usually found in quantum mechanics, for scattering of a wave on a potential V.
Naturally the (first-order) Born approximation is good only if the first correction is smaller than the incident field, and in general will be valid only if the series converges.

We should also note that the first Born approximation, where the wave is expressed as a sum of incident and diffracted secondary wave, the scattering of the secondary wave is neglected. So no multiple scattering is included in this approximation.

5 Rytov Approximation

The Rytov approximation is obtained by representing the total field as a complex phase:

$$\psi(\mathbf{r}) = e^{\phi(\mathbf{r})} \tag{5.1}$$
$$= e^{\phi_i(\mathbf{r})+\phi_s(\mathbf{r})} = \psi_i e^{\phi_s(\mathbf{r})} \ , \tag{5.2}$$

where we have also assumed that the total phase can be expressed as the sum of an incident and a scattered phase.
Then, from the Helmholtz equation for ψ we have:

$$\nabla^2 e^{\phi(\mathbf{r})} + k^2 e^{\phi(\mathbf{r})} = 0 \ . \tag{5.3}$$

Since

$$\nabla^2 e^{\phi(\mathbf{r})} = \nabla^2 \phi e^{\phi(\mathbf{r})} + (\nabla\phi)(\nabla\phi) e^{\phi(\mathbf{r})} \ ,$$

we get the following Riccati equation for the phase $\phi(\mathbf{r})$:

$$\nabla^2\phi + (\nabla\phi)\cdot(\nabla\phi) + k^2 = 0 \ . \tag{5.4}$$

Let us now again write the refractive index as $k(\mathbf{r}) = k_0 n(\mathbf{r})$ The field for $n(\mathbf{r}) = 1$, i.e. the field in a non-refractive medium, is of course the incident field $\psi_i(\mathbf{r})$, and its phase will satisfy

$$\nabla^2\phi_i + (\nabla\phi_i)\cdot(\nabla\phi_i) + k_0^2 = 0 \tag{5.5}$$

If we subtract (5.5) from (5.4), using $\phi = \phi_i + \phi_s$, we get

$$\nabla^2\phi_s + 2(\nabla\phi_i)\cdot(\nabla\phi_s) = -\left((\nabla\phi_s)\cdot(\nabla\phi_s) + k_0^2(n^2-1)\right) \ . \tag{5.6}$$

Now, using the identity

$$\nabla^2(\psi_i\phi_s) = (\nabla^2\psi_i)\phi_s + 2\psi_i(\nabla\phi_i)\cdot(\nabla\phi_s) + \psi_i\nabla^2\phi_s \ , \tag{5.7}$$

as well as $\nabla^2\psi_i = -k_0^2\psi_i$ and $\nabla\psi_i = (\nabla\phi_i)\psi_i$, we shall derive a wave equation involving the scattered phase ϕ_s, which is formally solvable using the free space Green's function.

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We shall first write (5.7) as

$$\nabla^2(\psi_i\phi_s) = -k_0^2\psi_i\phi_s + 2\psi_i(\nabla\phi_i)\cdot(\nabla\phi_s) + \psi_i\nabla^2\phi_s \ , \tag{5.8}$$

If we now multiply (5.6) by ψ_i, we can write (5.8) as

$$\nabla^2(\psi_i\phi_s) + k^2\psi_i\phi_s = \left((\nabla\phi_s)(\nabla\phi_s) + k_0^2(n^2-1)\right)\psi_i \ , \tag{5.9}$$

whose solution can be written as an integral using the free-space Green's function, to give:

$$\phi_s(\mathbf{r}) = \frac{1}{\psi_i(\mathbf{r})}\int G(\mathbf{r}-\mathbf{r}')\left[(\nabla\phi_s(\mathbf{r}'))\cdot(\nabla\phi_s(\mathbf{r}')) + k_0^2(n^2(\mathbf{r}')-1)\right]\psi_i(\mathbf{r}')d\mathbf{r}' \tag{5.10}$$

This equation is exact, but it's implicit and in practice provides no solution as it is. If we assume that the scattered phase ϕ_s is very small, then we can neglect $|\nabla\phi_s|^2$, and we obtain an approximate solution for the scattered phase

$$\phi_s(\mathbf{r}) \simeq \frac{1}{\psi_i(\mathbf{r})}\int G(\mathbf{r}-\mathbf{r}')[k_0^2(n^2(\mathbf{r}')-1)]\psi_i(\mathbf{r}')d\mathbf{r}' \tag{5.11}$$

The corresponding solution for the total field is then

$$\psi(\mathbf{r}) \simeq \psi_i(\mathbf{r})e^{\phi_s} \ . \tag{5.12}$$

This approximation is known as the (first) ***Rytov approximation***. It corresponds to taking the first order term in an infinite power series expansion of the phase $\phi(\mathbf{r})$. It is valid when $(\nabla\phi_s)^2 \ll k_0^2(n^2(\mathbf{r}')-1)$.

Only if you're interested (non-examinable):

It is interesting to compare the validity of the Born and Rytov approximations.

Note that the Born approximation can be seen as a Taylor series approximation of the field $\psi(\mathbf{r},\varepsilon)$ in powers of ε, where ε is a measure of the inhomogeneity. The Rytov approximation can also be seen as a Taylor series approximation of $\log\psi(\mathbf{r},\varepsilon)$ in powers of ε. In our case, ε was the space-dependent variation n_δ from a constant refractive index.

We shall reproduce here the analysis by Keller (see Keller J.B. 1969 *'Accuracy and validity of the Born and Rytov approximations'*, J. Opt Soc. Am. 59, 1003-04) and consider the one-dimensional case of a wave travelling in a inhomogeneous medium given by

$$\psi(x,\varepsilon) = e^{ik(\varepsilon)x} \ , \tag{5.13}$$

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and assume that $k(\varepsilon)$ is analytic in ε for $|\varepsilon|$ sufficiently small, so that it can be expanded in a power series in ε with coefficients $\mathbf{k}_j$:

$$k(\varepsilon) = \sum_{j=0}^{\infty} k_j \varepsilon^j \ . \tag{5.14}$$

The Born expansion gives

$$\psi(x,\varepsilon) = e^{ik_0x} \sum_{s=0}^{\infty} \varepsilon^s \sum_{l=0}^{s} \frac{(ix)^l}{l!} \sum_{j_1+\cdots+j_l=s} k_{j_1} \cdots k_{j_l} \tag{5.15}$$

The nth Born approximation $\psi_B^{(n)}(x,\varepsilon)$ is the sum of the first $n+1$ terms in the expression above:

$$\psi_B^{(n)}(x,\varepsilon) = e^{ik_0x} \sum_{s=0}^{n} \varepsilon^s \sum_{l=0}^{s} \frac{(ix)^l}{l!} \sum_{j_1+\cdots+j_l=s} k_{j_1} \cdots k_{j_l} \tag{5.16}$$

The Rytov expansion gives

$$\psi(x,\varepsilon) = e^{ik(\sum_{j=0}^{\infty} k_j \varepsilon^j)} \ , \tag{5.17}$$

and the nth Rytov approximation $\psi_R^{(n)}(x,\varepsilon)$ is obtained by taking the first $n+1$ terms in the sum in the exponent:

$$\psi(x,\varepsilon) = e^{ik(\sum_{j=0}^{n} k_j \varepsilon^j)} \ . \tag{5.18}$$

The size of the error of the nth Born approximation $\psi - \psi_B^{(n)}$ for small ε and large $|x|$ can be found by examining the coefficient of ε^{n+1} in (5.15). That coefficient contains a term proportional to x^{n+1}. So

$$\psi - \psi_B^{(n)} = e^{ik_0x} \mathrm{O}(\varepsilon^{n+1} x^{n+1}) \ . \tag{5.19}$$

Dividing this by ψ, and noting that ψ differs from e^{ik_0x} by terms of the order ε, we obtain for the relative error:

$$\frac{\psi - \psi_B^{(n)}}{\psi} = \mathrm{O}(\varepsilon^{n+1} x^{n+1}) \ . \tag{5.20}$$

The error for the nth Rytov approximation $\psi - \psi_R^{(n)}$ is:

$$\begin{aligned} psi - \psi_R^{(n)} &= e^{ik(\sum_{j=0}^{\infty} k_j \varepsilon^j)} - e^{ik(\sum_{j=0}^{n} k_j \varepsilon^j)} \\ &= \psi \left(1 - e^{-ik(\sum_{j=n+1}^{\infty} k_j \varepsilon^j)} \right) \\ &= \psi \mathrm{O}(\varepsilon^{n+1} x) \end{aligned}$$

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Dividing this by ψ gives for the relative error

$$\frac{\psi - \psi_R^{(n)}}{\psi} = \mathrm{O}(\varepsilon^{n+1} x) \ . \tag{5.21}$$

We can see then that the relative errors of the Born and the Rytov approximation are of the same order in the inhomogeneity parameter ε. However, the expressions obtained for the relative errors also show that they vary in a very different way as functions of x. For a single plane wave, the nth Rytov approximation is valid over a much larger range than is the nth Born approximation, however this advantage is lost for fields containing more than one wave, where the Rytov method must be applied to each wave separately and not to the total field ψ.

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Part II
Random Media

6 Statistics

We shall consider propagation and scattering in random media, i.e. media whose properties are described by randomly varying function of space (and in general also in time). These media shall be of two kinds:

• randomly rough surfaces, which shall be described in terms of a randomly varying surface height $h(x)$ (e.g. any surface of a real material, sea surface, terrain, which could be anything from forest canopy to urban),
• random extended media, which shall be described in terms of a randomly varying refractive index $n(x)$ (e.g. the atmosphere, the ocean, biological tissue or fluid in medical applications).

In either case, we shall model our system as a continuous stochastic process $\gamma(x)$ of a variable $x \in \mathbb{R}^n$, which will take a specific realisation with probability given by a **probability density function** (p.d.f.) $f(\gamma)$. (Here γ will be either surface height $h(x)$ or refractive index $n(x)$.) We can think of a specific realisation $\gamma(x)$ as a member of a given *ensemble* Ω of functions all having the same statistical nature. A single realisation will not have any practical meaning, and we need to consider averages over all possible realisations, i.e. **ensemble averages**, which shall be denoted by angled brackets: $< \cdot >$.
We therefore require a few statistical concepts and results to characterise our system. The necessary results are not extensive and some will already be well-known. Some familiarity with them is essential in the manipulation of the statistical quantities which arise, and we shall summarise them here.
Given a p.d.f. $f(\gamma)$, the **mean** of γ is

$$< \gamma >= \int \gamma' f(\gamma') d\gamma' \ . \tag{6.1}$$

Here $f(\gamma(x))d\gamma(x)$ is the probability that, at a point x, the function $\gamma(x)$ has a value in the uinterval $[\gamma, \gamma + d\gamma]$. We should note that generally the formal definition of an ensemble average involves the introduction of a probability measure P on $\mathbb{R}^n$, so the mean is defined by the *Riemann-Stieltjes* integral

$$< \gamma >= \int_\Omega \gamma dP \ .$$

We shall not worry about this, since the above Riemann-Stieljes integral reduces to an ordinary Riemann (or Lebesgue) integral if P has continuous

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derivative, which shall indeed be the case in all random media considered here.
The **variance** of γ is

$$< (\gamma - < \gamma >)^2 >= \int (\gamma - < \gamma >)^2 f(\gamma')d\gamma' \equiv \sigma^2 \ . \tag{6.2}$$

The **standard deviation** of γ is the square root of its variance:

$$\sigma = \sqrt{< (\gamma - < \gamma >)^2 >} \ . \tag{6.3}$$

A number of **assumptions** are usually made about the statistics, which are often for analytical convenience, but in most cases are also physically reasonable.

(1) $< \gamma >= 0$, i.e. the stochastic process is described by a random perturbation about a constant value, which we can then choose $= 0$.
This assumption is applicable to many real processes, for example most surfaces can be viewed as random variation over a flat surface and most continuous media can be characterised by random fluctuations of the refractive index about its value in free space. Nevertheless some quantities characterising the atmosphere, for example, such as temperature, will often gradually increase (or decrease) so that the resulting refractive index is a random fluctuation about a steady 'drift'.
Note that with this assumption the standard deviation is just the root mean square (r.m.s.) of γ: $\sigma = \sqrt{< \gamma >^2}$.

(2) The random variable are normally distributed, so the p.d.f. is a Gaussian:

$$f(\gamma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\gamma^2/2\sigma^2} \ . \tag{6.4}$$

This assumption is often physically reasonable. For example many rough surfaces arise as the result of a large number of independent random events and are therefore normal by the Central Limit Theorem. However, it is wrong for important cases such as the sea surface. (The sea typically has sharper peaks than troughs, so the height distribution is not symmetric about the mean, as would be required by the symmetry of the normal distribution about the origin.)

For normal random variables we have the following:
If γ is normal, then so is $\gamma(x_1) + \gamma(x_2)$, and $\int \gamma(x)dx$ over any interval.
All the one-point statistics are determined by the mean $< \gamma >$ and variance $< \gamma^2 >$. For example we have $< \gamma^{2n+1}(x) >= 0$ for all n, and

$$< \gamma^4(x) >= 3\sigma^2 < \gamma^2 > \tag{6.5}$$

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This can be seen by writing

$$< \gamma^n >= \int \gamma'^n f(\gamma') dh'$$

and integrating by parts, noting that in the case where $f(\gamma)$ is Gaussian $hf(\gamma) = -\sigma^2 \frac{d}{d\gamma}(f(\gamma))$.

The quantities introduced so far give us only partial information about the stochastic process $\gamma(x)$. In particular, the p.d.f. gives the probability that $\gamma(x)$ has a particular value at x, but tells us nothing about how the value at that point may depend from values at other points.
This is provided by the **autocorrelation function** (a.c.f.) $\rho(x_1, x_2)$

$$\rho(x_1, x_2) =< (\gamma(x_1) - < \gamma(x_1) >)(\gamma(x_2) - < \gamma(x_2) >) > , \qquad (6.6)$$

which reduces to

$$\rho(x_1, x_2) =< \gamma(x_1)\gamma(x_2) > \qquad (6.7)$$

when $< \gamma(x) >= 0$.
(Note that in the literature the a.c.f. is often defined by the ***normalised version*** of the above:

$$\rho(x_1, x_2) = \frac{< \gamma(x_1)\gamma(x_2) >}{\sigma^2}$$

and (6.7) is referred to as ***covariance***.)
We shall only consider **statistically stationary** stochastic processes, i.e. those for which all statistics are translationally invariant. In this case the a.c.f. only depends on the distance $| x_2 - x_1 |= \xi$, and we can write:

$$\rho(\xi) =< \gamma(x)\gamma(x + \xi) > \qquad (6.8)$$

Associated with the a.c.f., it is useful to define a **correlation length** L as the value of separation ξ at which $\rho(\xi) = e^{-1}\rho(0)$. The correlation length gives an indication of how far one need to be from a given point for the value of γ at that point to be negligible. Together with the wavelength of the acoustic field and other experimental parameters, it needs to be taken into account when determining parameters in a numerical solution and the size of the necessarily finite experimental sample.

The a.c.f. is clearly a very general measure with which to characterise a stochastic process, since it determines both the correlation length and the standard deviation $\sigma = \sqrt{< \gamma >^2} = \sqrt{\rho(0)}$. It provides information about the spatial variation of the random variable in our stochastic process, but is

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not related to the probability distribution of the random process. The a.c.f. can have various forms depending on the nature of the irregularities, and they will in many cases be different from the p.d.f. of the stochastic process.
Examples:
(a) $\rho(\xi) = \sigma^2 e^{-\xi^2/L^2}$
(b) $\rho(\xi) = \sigma^2 e^{-|\xi|/L}$
(c) $\rho(\xi) = \sigma^2(1 + |\xi|)e^{-|\xi|/a}$

The Gaussian a.c.f. (a) is widely used in a variety of contexts. (b) is often used to characterise fractal surfaces. The fourth order power law a.c.f. (c) often occurs in the context of turbulence. We can assume that ρ is an even function, and falls from its maximum σ at $\xi = 0$ to zero at large $|\xi|$. Unlike (b), the functions (a) and (c) are smooth at the origin, i.e. $d\rho/d\xi = 0$ at $\xi = 0$. If used to characterize surfaces, then, they would describe a surface that, 'under a microscope', would appear smooth.

Another function, used particularly in the context of turbulent random media, is the **structure function** $D(x_1, x_2)$, defined by

$$D(x_1, x_2) =< (\gamma(x_1) - \gamma(x_2))^2 > \quad . \tag{6.9}$$

The structure function is independent of the mean $< \gamma(x) >$ so is particularly useful for describing processes that do not have constant mean, where, for slowly varying mean, i.e. processes for which the difference $\gamma(x + \xi) - \gamma(x)$ is not affected by the change in the mean (γ is a random function with *stationary increments*), we can write

$$D(\xi) =< (\gamma(x + \xi) - \gamma(x))^2 > \quad . \tag{6.10}$$

In the fully stationary case, when we have $< \gamma(x) >= 0$, then the structure function is related to the autocorrelation function as follows:

$$D(\xi) = 2(\rho(0) - \rho(\xi)) \ . \tag{6.11}$$

There is a another important consequence of stationarity.
For stationary random functions $\gamma(x)$, there exists an integral expansion similar to the Fourier expansion for non-random functions. Using this expansion, it is possible to show (***Wiener-Khinchin theorem***) that one can write

$$S(\nu) = \int_{-\infty}^{\infty} \rho(\xi) e^{i\xi\nu} d\xi \tag{6.12}$$

The function $S(\nu)$ is called the **power spectrum**, and forms a Fourier transform pair with the autocorrelation function $\rho(\xi)$. We shall see later that the power spectrum is related to the intensity of the scattered field.

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A further important (and very useful) consequence of stationarity is the **ergodic theorem**, which states that

Space (time) averages = ensemble averages

So we have

$$< \gamma(x)\gamma(x+\xi) >= \lim_{A\to\infty} \frac{1}{A} \int_A \gamma(x)\gamma(x+\xi)dx \ , \tag{6.13}$$

where A is the volume of our space.
The equivalent equation for a stationary time-dependent stochastic process, which is more usually found in the literature, is

$$< \gamma(x,t)\gamma(x+\xi,t) >= \lim_{T\to\infty} \frac{1}{T} \int_0^T \gamma(x,t)\gamma(x+\xi,t)dt \tag{6.14}$$

If we use (6.13) for the a.c.f. in the power spectrum (6.12), we obtain

$$S(\nu) = \lim_{A\to\infty} \frac{1}{A} \left| \int_{-\infty}^{\infty} \gamma(\mathbf{x}) e^{\nu\cdot\mathbf{x}} d\mathbf{x} \right|^2 \ , \tag{6.15}$$

and therefore

$$\int_{-\infty}^{\infty} S(\nu)d\nu = \sigma^2. \tag{6.16}$$

The above is a special case of the general result that moments of the power spectrum give r.m.s. averages of higher order derivatives of the stochastic process:

$$\int_{-\infty}^{\infty} S(\nu)\nu^{2n}d\nu = \left\langle \left(\frac{\partial^n \gamma}{\partial x^n} \right)^2 \right\rangle \tag{6.17}$$

Finally in this section, we list some further basic properties or rules for averaging which will come in useful in the calculations we need to carry out, involving integrals and derivatives of γ.

(1) If $F(x)$ is a *deterministic* function, and $A(\gamma)$ is any functional of γ, then

$$\left\langle \int A(\gamma(x))F(x)dx \right\rangle = \int \langle A(\gamma(x)) \rangle F(x)dx$$

This follows by linearity of the integral.

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(2) A function which sometimes arises is the average of the product of γ and its derivative:

$$\left\langle \gamma(y)\frac{d\gamma(x)}{dx}\right\rangle = \left.\frac{d\rho}{d\xi}\right|_{\xi=y-x}.$$

In order to prove (2), write

$$\gamma(y)\gamma'(x) = \gamma(y)\ \lim_{\epsilon\to 0}\frac{1}{\epsilon}\left[\gamma(x+\epsilon)-\gamma(x)\right]$$

The result follows by averaging the right-hand-side and taking the average inside the limit sign.

Numerical generation of random surfaces *(Exercise)*

It is instructive in the manipulation of averages to consider how a continuous rough surface $h(x)$ may be simulated. The simplest method is to represent $h(x)$ as a sum of sinusoidal components as follows:

Suppose we wish to represent an example of a surface with a given a.c.f. $\rho(\xi)$. The basic steps are:

(1) Define $A(\nu) = \sqrt{B(\nu)}$ where B is the cosine transform of ρ,

$$B(\nu) = \frac{2}{\pi}\int_{-\infty}^{\infty}\rho(\xi)\cos(\xi\nu)\ d\xi.$$

(We can assume that $B(\nu)$ has compact support.)

(2) Choose some number N of equally-spaced frequencies $\nu_j = j\Delta\nu$, say, where N and ν_N are large enough to resolve the features of B adequately.

(3) Choose N *independent random phases* ϕ_j, uniformly in $[0, 2\pi)$.

(4) Define a function $h(x)$ by

$$h(x) = \sqrt{\Delta\nu}\ \sum_{n=1}^{N} A_n \sin(\nu_n x + \phi_n),$$

where $A_n = A(\nu_n)$. Then h is a continuous function of x with the required statistics, as we can show. The random part of this definition is in the choice of random phases (3). Each different set of phases gives rise to a new realisation of a random process h, and averages can therefore be taken over this ensemble.

First, it is easy to check that $< h >= 0$, and for large N the values $h(x)$ are normally distributed by the central limit theorem. To calculate the a.c.f. of h, first write $x_n = \nu_n x + \phi_n$, and $y_n = \nu_n y + \phi_n$. Then since ϕ_n is uniform in $[0, 2\pi)$, it is easy to show for example that

$$
\begin{aligned}
< \sin x_n > &= 0 \\
< \sin x_n \cos x_n > &= 0 \\
< \sin^2 x_n > &= 1/2 \\
< \sin x_n \sin y_n > &= \frac{1}{2}\cos(\nu_n \xi)
\end{aligned}
$$

where $\xi = y - x$.
So the a.c.f. can be written

$$
\begin{aligned}
< h(x)h(y) > &= \Delta\nu \sum_{m,n=1}^{N} A_m A_n < \sin x_n \sin y_m > \\
&= \Delta\nu \sum_{n=1}^{N} A_n^2 < \sin x_n \sin y_n > \\
&= \frac{\Delta\nu}{2} \sum_{n=1}^{N} A_n^2 \cos(\nu_n \xi) \\
&\cong \int_{-\infty}^{\infty} B(\nu)\cos(\nu\xi)\ d\nu \\
&= \rho(\xi)
\end{aligned}
$$

as required. Here we have used the fact that $\sin x_n$ and $\sin y_m$ are independent.

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7 Scattering from randomly rough surfaces

7.1 Rayleigh criterion

The scattering of plane waves from a flat boundary between two media is a typical canonical problem, where analytical solutions are straightforward and well-known. It is an idealized case: all real surfaces are rough. The scattering problem will then depend on the 'roughness' of the surface, and exact analytical solutions will not be generally available. In this chapter we shall look at ways of characterizing the surface, and consider some approximate solutions.

Suppose then that a time-harmonic plane wave

$$\psi_i = \exp(ik[x \sin\theta - z\cos\theta])$$

is incident on a boundary which is now an irregular function of position. (We suppress above and in what follows the harmonic time dependence). We will assume here that the *surface normal* is well-defined and continuous everywhere along the boundary. One of the earliest treatments of the rough surface problem was by Rayleigh (1907), who considered the phase change due to height differences in the case when the wavelength is small compared with the horizontal scale of surface variation.

Calculating the phase difference $\Delta\phi$ between wavefronts along two specularly reflected rays as in the schematic diagram gives

$$\Delta\phi = 2k(h_2 - h_1)\cos\theta$$

where h_1, h_2 are the heights at the two points of incidence. The interference between these two rays depends on the magnitude of $\Delta\phi$ with respect to π. When the surface is nearly flat, $\Delta\phi \ll \pi$ and the two rays are in phase (so interfere constructively), but for large deviations we may have $\Delta\phi \sim \pi$, giving destructive interference. This lead to the so-called Rayleigh criterion for distinguishing different roughness scales, by which surfaces may be called 'rough' or 'smooth' according to whether $\Delta\phi$ greater than or less than $\pi/2$.

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If this is averaged across the surface, then $(h_2 - h_1)$ may be replaced by the average r.m.s. surface height σ, which gives the surface r.m.s. deviation from a flat surface, and is defined by $\sigma^2 =< h^2(x) >$. The Rayleigh criterion for 'smoothness' is then expressed by

$$k\sigma \cos\theta < \frac{\pi}{4} \ . \tag{7.1}$$

The quantity $k\sigma \cos\theta$ is referred to as the ***Rayleigh parameter***. Note that this is dependent on angle of incidence, and implies that all surfaces become 'smooth' for low grazing angles. At optical wavelengths this is often reasonable, but is less true, for example, for typical radar wavelengths of 3cm or whenever the roughness length scale becomes comparable to a wavelength. In that case the Rayleigh criterion fails to take into account 'multiple scattering' effects such as shadowing and diffraction.

We shall consider here two of the most common and fundamental approximate methods for calculating scattering from rough surfaces, and derive expressions for some of the quantities with which it is possible to characterise the scattered field. These will necessarily be mostly statistical averages, in general 'moments' of the scattered field ψ_s, such as for example the **mean (coherent) field** $< \psi_s >$, the **field coherence function** $m(y-x) = < \psi_s(x)\psi*_s(y) >$ (so that $m(0)$ is the **mean intensity** of the scattered field, and higher moments and related functions.

Throughout, we shall use the following notation and assumptions:

The randomly rough surface will be characterised by a function $h(x)$. We will consider h to be a member of a statistical ensemble, which is stationary with respect to translation in x, with mean square height $< h^2 >= \sigma^2$, autocorrelation function $\rho(\xi)$, and we shall take the constant mean $< h >$ to be 0.

We shall usually take the incident field to be a plane wave

$$\psi_i(x,z) = e^{i\mathbf{k}\cdot\mathbf{r}} = e^{ik(x\sin\theta - z\cos\theta)}$$

(and the equivalent in 3D), which still allows for our results to be applicable in more general cases when using linear equations, since any wave can be written as a superposition of plane waves.
If we then define the Fourier transform $\hat{\psi}(\nu)$ of the scattered field ψ_s along the horizontal mean plane, $z = 0$.

$$\hat{\psi}(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_s(x,0)e^{-i\nu x} \, dx \ , \tag{7.2}$$

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we can then write the scattered field at a point (x, z) as a superposition of Fourier components. Each Fourier component at the mean plane, $\hat{\psi}(\nu)$, will be scattered away from the surface $z = 0$ as another plane wave

$$\hat{\psi}(\nu)e^{iqz}$$

satisfying the Helmholtz equation. This gives $q = \sqrt{k^2 - \nu^2}$, where we have taken the positive (or positive imaginary) root to ensure that the scattered field consists of outgoing waves.
The scattered field at a point (x, z) in the medium can therefore be written

$$\psi_s(x, z) = \int_{-\infty}^{\infty} \hat{\psi}(\nu)e^{i(\nu x+qz)}d\nu \tag{7.3}$$

7.2 Approximation for small surface height $k\sigma \ll 1$

In this case **perturbation theory** can be applied. The method is essentially to expand quantities appearing in the problem that are functions of surface height, in order to form a simpler boundary problem on the mean plane, i.e. on $z =< h(x) >= 0$.
We seek the solution for the scattered field ψ_s and its mean $< \psi_s >$. Suppose that the surface obeys the Dirichlet condition, $\psi(x, h) = 0$. We proceed as follows:

(1) Expand the boundary condition to order h. Thus we obtain

$$\psi_i(x, 0) + \psi_s(x, 0) + h(x)\left(\frac{\partial \psi_i}{\partial z} + \frac{\partial \psi_s}{\partial z}\right) = 0 \ + O(h^2) \tag{7.4}$$

using $\psi = \psi_i + \psi_s$. Here and below, unless specified otherwise, the functions are to be evaluated on the mean plane $z = 0$.

(2) Next, assume that the scattered field everywhere can be expanded in powers of kh, say

$$\psi_s(x, z) = \psi_0(x, z) + \psi_1(x, z) + \psi_2(x, z) + ... \tag{7.5}$$

where ψ_n is of order $O(h^n)$ for all n, so that ψ_0 is the known, deterministic *flat surface* reflected field, and ψ_n is stochastic for $n \geq 1$ since it depends on the specific choice of surface $h(x)$.

(3) Now truncate (7.5) at $O(h)$, substitute into (7.4), and neglect terms of order $O(h^2)$. This gives an *approximate boundary condition* which holds on the mean plane

$$\psi_i + \psi_0 + \psi_1 + h(x)\left(\frac{\partial \psi_i}{\partial z} + \frac{\partial \psi_0}{\partial z}\right) = 0 \tag{7.6}$$

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where again all functions are evaluated at points $(x, 0)$. In this equation the third term ψ_1 is the only unknown component, since the remaining functions are the zero order (flat surface) forms, so we have an explicit approximation to the solution along the mean plane.
The first two terms in (7.6) cancel, since they represent the total field which would exist in the case of a flat surface, which vanishes by the Dirichlet boundary condition. We can now equate terms of equal order. Equating $O(h)$ (first order) terms gives

$$\psi_1 = -h(x)\left.\frac{\partial(\psi_i + \psi_0)}{\partial z}\right|_{(x,0)}$$

which gives

$$\psi_1(x, 0) = -2h(x)\frac{\partial \psi_i}{\partial z}. \tag{7.7}$$

This solves for ψ_1 explicitly on the mean plane. From this we can obtain the scattered field everywhere to $O(h)$, using $\psi_s = \psi_0 + \psi_1 + O(h^2)$. Once ψ_1 is known on any plane we can split it into Fourier components, and propagate these outwards (using radiation conditions to determine the direction):
Consider in particular the case of an incident plane wave, $\psi_i = e^{ik(x\sin\theta - z\cos\theta)}$. We then have

$$\psi_1(x, 0) = 2h(x)ik\cos\theta\; e^{ikx\sin\theta}. \tag{7.8}$$

Denote by $\hat{h}$ the transform of h,

$$\hat{h}(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)e^{-i\nu x}\; dx\;,$$

and similarly by $\hat{\psi}_0$ the transform of $\psi)_0$. Then, from (7.2) and (7.8) we get

$$\hat{\psi}(\nu) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_s(x, 0)e^{-i\nu x}\; dx = ik\frac{\cos\theta}{\pi}\hat{h}(\nu - k\sin\theta) + \hat{\psi}_0\;, \tag{7.9}$$

so that, from (7.3), taking the inverse Fourier transform and since $\psi_s = \psi_0 + \psi_1$,

$$\psi_1(x, z) = ik\frac{\cos\theta}{\pi}\int_{-\infty}^{\infty} \hat{h}(\nu - k\sin\theta)\; e^{i(\nu x + qz)}\; d\nu \tag{7.10}$$

where as before $q = \sqrt{k^2 - \nu^2}$.
We note that the formulation of the solution using perturbation theory in the approximation of small height depends on the boundary conditions. In particular, it will be different for Neumann and for impedance boundary conditions, although some results are applicable in general.

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Averaging:
The dependence of the field on the surface is now clear to first order in surface height. Taking the average of (7.10) immediately gives the mean of this perturbation as

$$< \psi_1 >= 0$$

everywhere, since $< h(x) >= 0$, so that *first order perturbation theory predicts no change in the coherent field.* (Equivalently, the effective reflection coefficient is the same to first order as the flat surface coefficient.) Although we have examined the Dirichlet condition, this holds for arbitrary boundary conditions since the first order term is always linear in the boundary itself.

Angular spectrum:
Now consider the angular spectrum to find the scattered energy. For a plane wave incident at angle θ on a given surface, the far-field intensity in the transform space is given by $I_\theta(\nu) = |\hat{\psi}(\nu)|^2$, so from (7.2) and (7.8), we can write its average as

$$\left\langle |\hat{\psi}(\nu)|^2 \right\rangle = \left\langle \frac{k^2 \cos^2\theta}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x)h(x')e^{i(k\sin\theta-\nu)(x-x')}\, dx'\, dx \right\rangle . \tag{7.11}$$

Making the change of variables $\xi = (x - x')$, $X = (x + x')$, this becomes

$$\begin{aligned} \left\langle |\hat{\psi}(\nu)|^2 \right\rangle &= \frac{k^2 \cos^2\theta}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-i\nu X} \rho(\xi) e^{i(k\sin\theta-\nu)\xi}\, d\xi\, dX \\ &= 2\frac{\delta(\nu)}{\pi}\, k^2 \cos^2\theta\, S(k\sin\theta - \nu) \end{aligned} \tag{7.12}$$

where S is again the power spectrum of the surface and δ is the delta-function. Since the averaged scattered intensity is non-zero, this approximation to first order does lead to a contribution to the diffusely scattered field

$$\psi_d = \psi_{sc} - < \psi_{sc} > ,$$

even though it predicts no change in the coherent field, as we saw above. First order perturbation theory therefore does not obey conservation of energy.

7.3 Tangent plane approximation

The approximation considered in this section is often presented as a *small slope* approximation, valid for $< |dh/dx| >\ll 1$. Quantifying the approximation in this way, relative to the surface slope, can indeed be useful. It is nevertheless important to keep in mind that this is only meaningful when the surface height is expressed in units of wavelength. Also, as we shall see later, the quantity that really needs to be small is the surface ***curvature***, i.e. we need to assume that at every point on the surface the radius of curvature is large with respect to the wavelength of the incident field.
We shall use here the integral form of the wave equation (2.13), so the scattered field at $\mathbf{r}$ is given by

$$\psi_{sc}(\mathbf{r}) = \int_S \psi(\mathbf{r}_0)\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - G(\mathbf{r},\mathbf{r}_0)\frac{\partial \psi}{\partial n}(\mathbf{r}_0)d\mathbf{r}_0\ , \tag{7.13}$$

where $\mathbf{r}_0$ is on the surface and ψ and $\partial\psi/\partial n$ are unknown. We note that the use of this integral form implies integration over a closed surface, however we can achieve this by 'mathematically closing' the surface by means either of a surface at infinity, or a surface 'behind' the given rough surface and infinitesimally close to it. If we have a finite surface (as of course is the case in all real applications), the first choice ignores the finite surface size, whilst the second will include contributions coming from the edges.
The unknowns are approximated by using the **Kirchhoff approximation** (sometimes referred to as the tangent plane, or the geometrical optics solution), which treats any point on the scattering surface as though it were part of an infinite plane, parallel to the local surface tangent. We make the following assumptions:

(1) that the surface can be treated as 'locally flat';

(2) and that the incoming field at each point is just ψ_i.

The second assumption neglects multiple scattering, which can give rise to secondary illumination of any point on the surface.
Consider for simplicity the Dirichlet boundary condition, so that we are solving the integral equation

$$\psi_{sc}(\mathbf{r}_s) = -\int_S G(\mathbf{r},\mathbf{r}_0)\frac{\partial \psi}{\partial n}(\mathbf{r}_0)d\mathbf{r}_0\ . \tag{7.14}$$

Under the assumptions above, we can approximate $\partial\psi/\partial n$ at each point by the value it would take for a flat surface with slope dh/dx:

$$\frac{\partial \psi}{\partial n} \cong 2\frac{\partial \psi_i}{\partial n}. \tag{7.15}$$

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This neglects curvature and shadowing by other parts of the surface. The field then becomes

$$\psi_s(\mathbf{r}) = -2\int G(\mathbf{r},\mathbf{r}_0)\frac{\partial\psi_i}{\partial n}(\mathbf{r}_0)d\mathbf{r}_0. \tag{7.16}$$

Similar formulae are easily obtained for Neumann condition and more generally an interface between two media.
When the surface is not perfectly reflecting, the normal derivative of the field at the surface will be given by

$$\frac{\partial\psi}{\partial n} \cong (1 - R(\mathbf{r}_0))\frac{\partial\psi_i}{\partial n}\ , \tag{7.17}$$

where $R(\mathbf{r}_0)$ is the flat surface reflection coefficient; and the field at the surface by:

$$\psi \cong (1 + R(\mathbf{r}_0))\psi_i\ . \tag{7.18}$$

If we further consider the *far-field approximation*, we can approximate the argument of the free space Green's function, $k|\mathbf{r}-\mathbf{r}_0|$ by

$$k|\mathbf{r}-\mathbf{r}_0| \cong kr - k\hat{\mathbf{r}}\cdot\mathbf{r}_0\ , \tag{7.19}$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of observation $\mathbf{r}$. The derivative of the Green's function can then be approximated by

$$\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} \cong -\frac{ie^{ikr}}{4\pi r}(\mathbf{n}\cdot\mathbf{k}_{sc})e^{-i\mathbf{k}_{sc}\cdot\mathbf{r}_0}\ , \tag{7.20}$$

where $\mathbf{k}_{sc} = k\hat{\mathbf{r}}$ is the wavevector of the scattered wave. Using these approximations in equation (7.13), we obtain for the scattered field

$$\psi_{sc}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r}\int_S((R\mathbf{k}^- - \mathbf{k}^+)\cdot\mathbf{n})e^{-i\mathbf{k}^-\cdot\mathbf{r}_0}d\mathbf{r}_0\ , \tag{7.21}$$

where

$$\begin{aligned}
\mathbf{k}^- &= \mathbf{k}_i - \mathbf{k}_{sc}\\
\mathbf{k}^+ &= \mathbf{k}_i + \mathbf{k}_{sc}\ .
\end{aligned}$$

If θ_1 is the angle of incidence (measured from the normal), and θ_2 and θ_3 are, respectively, the angle of the scattered wave with the normal, and the angle of the scattered wave with the x-axis in the plane (x, y), then

$$\begin{aligned}
\mathbf{k}_i &= k(\hat{\mathbf{x}}\sin\theta_1 - \hat{\mathbf{z}}\cos\theta_1)\\
\mathbf{k}_{sc} &= k(\hat{\mathbf{x}}\sin\theta_2\cos\theta_3 + \hat{\mathbf{y}}\sin\theta_2\sin\theta_3 + \hat{\mathbf{z}}\cos\theta_2)\ .
\end{aligned}$$

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We can now convert the integration in equation (7.21) to integration over the mean plane of the surface, S_M, by noting that an area element of the rough surface, $d\mathbf{r}_0$, projects onto the mean plane of an area element of the mean plane $d\mathbf{r}_M$, with the area elements related by

$$\mathbf{n}dr_0 \cong \left(-\hat{\mathbf{x}}\frac{\partial h}{\partial x_0} - \hat{\mathbf{y}}\frac{\partial h}{\partial y_0} + \mathbf{z}\right) dr_M \ . \tag{7.22}$$

The scattered field can therefore be written in the general form

$$\psi_{sc}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r}\int_{S_M}\left(a\frac{\partial h}{\partial x_0} + b\frac{\partial h}{\partial y_0} - c\right) e^{ik(Ax_0+By_0+Ch(x_0,y_0))}dx_0dy_0 \ , \tag{7.23}$$

where

$$\begin{aligned} A &= \sin\theta_1 - \sin\theta_2\cos\theta_3 \\ B &= -\sin\theta_2\sin\theta_3 \\ C &= -(\cos\theta_1 + \cos\theta_2) \ ; \end{aligned} \tag{7.24}$$

and

$$\begin{aligned} a &= \sin\theta_1(1-R) + \sin\theta_2\cos\theta_3(1+R) \\ b &= \sin\theta_2\sin\theta_3(1+R) \\ c &= \cos\theta_2(1+R) - \cos\theta_1(1-R) \ . \end{aligned} \tag{7.25}$$

Note that this approximation for the scattered field has been derived within the far-field approximation, and for an incident plane wave. In order to make analytical manipulations possible, further approximations are usually made. In general, the reflection coefficient is a function of position on the surface. We shall assume instead that R *is constant*. With this approximation, and for $C \neq 0$, we can eliminate the terms involving partial derivatives of the surface by performing a partial integration. Carrying out the integration with the assumption of independent integration limits for x_0 and y_0, and taking the surface to be of finite extent, defined by $-X \leq x_0 \leq X$ and $-Y \leq y_0 \leq Y$, gives a scattered field of the form

$$\psi_{sc}(\mathbf{r}) = -\frac{ie^{ikr}}{4\pi r}2F(\theta_1,\theta_2,\theta_3)\int_{S_M} e^{ik\phi(x_0,y_0)}dx_0dy_0 + \psi_e \ , \tag{7.26}$$

where the phase function $\phi(x_0, y_0)$ is

$$\phi(x_0,y_0) = Ax_0 + By_0 + Ch(x_0,y_0) \ , \tag{7.27}$$

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the angular factor $F(\theta_1, \theta_2, \theta_3)$ is

$$F(\theta_1, \theta_2, \theta_3) = \frac{1}{2}\left(\frac{Aa}{C} + \frac{Bb}{C} + c\right) \ , \tag{7.28}$$

and the term ψ_e is given by

$$\begin{aligned}\psi_e(\mathbf{r}) \quad &= \quad -\frac{ie^{ikr}}{4\pi r}\left[\frac{ia}{kC}\int \left(e^{ik\phi(X,y_0)} - e^{ik\phi(-X,y_0)}\right) dy_0\right. \\ &+ \quad \left.\frac{ib}{kC}\int \left(e^{ik\phi(x_0,Y)} - e^{ik\phi(x_0,-Y)}\right) dx_0\right] \end{aligned} \tag{7.29}$$

In the above approximation the angular factor depends on the boundary conditions. The term ψ_e is often referred to as 'edge effects', since it involves the values of the phase function at the surface edges.

We can now calculate average quantities of the scattered field, when $h(x, y)$ is a random surface with some probability density $f(h)$. The average of the scattered field, i.e. the **coherent field** is given by

$$\langle\psi_{sc}(\mathbf{r})\rangle = -\frac{ie^{ikr}}{4\pi r} 2F \int_{S_M}\int_{-\infty}^{\infty} e^{ik\phi(x_0,y_0)} f(h) dh dx_0 dy_0 \ . \tag{7.30}$$

Assuming stationarity, and using the explicit expression for the phase function given by equation (7.27), we obtain

$$\langle\psi_{sc}(\mathbf{r})\rangle = -\frac{ie^{ikr}}{4\pi r} 2F\hat{f}(kC) \int_{S_M} e^{ik(Ax_0+By_0)} dx_0 dy_0 \ , \tag{7.31}$$

where $\hat{f}(kC)$ is the Fourier transform of the probability density function, with respect to the transform variable kC.
The average of the intensity, or of the **angular spectrum**, of the diffuse field $\psi_d = \psi_{sc} - < \psi_{sc} >$ is given by

$$\left\langle|\psi_d|^2\right\rangle = \langle\psi_{sc}\psi{*}_{sc}\rangle - \langle\psi_{sc}\rangle\langle\psi{*}_{sc}\rangle \ . \tag{7.32}$$

This expression is far more complicated than the equivalent one obtained in the 'small height' approximation, because the coherent field is now different from zero. Further approximations will be necessary to obtain an expression of practical use for the angular spectrum in the Kirchoff approximation.

8 Wave Propagation through Random Media

References:

A. Ishimaru, Wave Propagation and Scattering in Random Media
B.J. Uscinski, Elements of Wave Propagation in Random Media

Remarks

This section concerns waves scattered by randomness or irregularities in the medium through which they are propagating. In many situations the wave speed varies randomly, for example in the atmosphere or the ocean. Sometimes this variation may be highly localized, such as a patch of turbulent air (e.g. over a hot road) or bathroom glass. These effects cause *focusing*, somewhat like that of a lens, and produce regions of both high and low intensity. (Familiar examples include the twinkling of stars, or the pattern of light in a swimming pool.)

There are essentially two mechanisms which contribute to these effects, and we shall refer to them as:

(i) *diffraction* (distance effect): i.e. the evolution of an irregular wave beyond a fixed plane. This allows focusing of rays as in a lens even when the medium is homogeneous; and

(ii) *scattering*, i.e. the continuous evolution of phase with propagation due to extended irregularities, causing bending of rays.

We will consider these mechanisms only for *weakly scattering media*. Roughly speaking, 'weak scattering' corresponds to small angles of scatter, so that a plane wave may become scattered into a narrow range of directions close to the original direction. This allows us to use the *parabolic wave equation*, which was derived in Lecture 3 as a small angle approximation. We will assume throughout this section that the parabolic equation holds, and that there is a definite predominant direction of propagation (which can be taken to be horizontal).

In an extended medium the effects (i) and (ii) mentioned above of course occur simultaneously, but we shall see that it is possible to treat them separately under reasonable assumptions.

We shall first consider the case in which the random irregularities occur within a thin layer.

8.1 Propagation beyond a thin phase screen

Suppose that we have initially a plane wave $\psi = e^{ikx}$ of unit amplitude propagating horizontally, so that the *reduced wave*, i.e. ψe^{-ikx}, is just $E(x, z) \equiv 1$.

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8.1 Propagation beyond a thin phase screen

Suppose that E encounters a thin vertical layer in the region $x \in [-\xi, 0]$, say, in which the wave speed $c(z)$ is slightly irregular. (This may represent for example a jet of hot air, or a turbulent layer.)

Denote the **refractive index** $n(z) = c_0/c(z)$, where c_0 is the background or free wave speed, and write

$$n(z) = 1 + w(z) \ . \tag{8.1}$$

We will assume that

- the function $w(z)$ is small: $w(z) \ll 1$,
- $w(z)$ is a continuous random fluctuation, with the following properties:
  - it has mean zero, i.e. $< w(z) >= 0$ for all z,
  - it is stationary in z, so, e.g. $< w(z_1)w(z_2) >=< w(z)w(z+\xi) >$,
  - it is normally distributed, i.e. its probability distribution function is Gaussian.

Initial effect: In the assumption of weak scattering and for a thin enough layer, the field will only suffer a phase change on going through the layer. If a wave has wavenumber k before entering the layer, the wavenumber in the layer will be given by $kn(z) = k + kw(z)$, and the reduced wave will acquire a phase

$$\phi(z) = k\xi w(z), \tag{8.2}$$

where ξ is the thickness of the layer.

Then E emerges from the layer with a pure phase change,

$$E(0, z) \ = \ e^{i\phi(z)} \tag{8.3}$$

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Evolution of the field and the moment equations

We shall use the thin screen model described above, and the parabolic equation which applies under the assumptions we made, to derive evolution equations for the **moments of the field**, in particular the **first moment** or mean field

$$m_1(x) \;=\; \langle E(x,z)\rangle\,. \tag{8.4}$$

(note this is a function of x only, by stationarity)
and the **second moment** (transverse autocorrelation) of the field, defined as

$$m_2(x,\eta) \;=\; \langle E(x,z)E^*(x,z+\eta)\rangle \tag{8.5}$$

so that the mean of the intensity $I(x,z) = |E|^2$ can be written $< I(x) >= m_2(x,0)$.
Evolution equations will be derived first just for propagation beyond a thin screen, as an introduction to the concept, then for propagation in an extended random medium, for some of the moments of the field: the *moment equations*.

It is a primary aim of the study of random media to examine the evolution of the field E with distance beyond a layer and find its statistics. There are many reasons for this requirement: for example in ocean acoustics one can almost never know the refractive index in detail, but statistical information can help overcome communications and navigational problems, or may be used for remote sensing of the environment. In other situations the measurement devices themselves may be detecting time or spatial averages.
Suppose for example we wish to find the mean intensity of the field. For a given medium it will not be possible to obtain a general solution for the wavefield or its intensity as a function of position. However, some of the statistical moments, such as field autocorrelation, themselves obey evolution equations which take a relatively simple form since the fluctuations in the medium have been 'averaged out', and can be solved or their solutions approximated analytically.

Before deriving equations for the evolution of the first and second moments, we shall make some heuristic remarks.
As the field evolves, the pure phase fluctuations which are imposed initially, equation (8.3), become converted to amplitude variations. (In terms of ray theory, this happens as the layer focuses or de-focuses the rays passing through it, and the intensity changes with the ray density.)
This can be shown and quantified roughly as follows:

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At a small distance x beyond the layer, we can take a Taylor expansion of the field $E(x,z)$ about $E(0,z) = e^{i\phi(z)}$:

$$E(x,z) \simeq E(0,z) + x\left.\frac{\partial E(x,z)}{\partial x}\right|_{x=0} + \frac{x^2}{2}\left.\frac{\partial^2 E(x,z)}{\partial x^2}\right|_{x=0} .$$

If we then use (8.3) and the parabolic wave equation

$$\frac{\partial E}{\partial x} = \frac{i}{2k}\frac{\partial^2 E}{\partial z^2}. \tag{8.6}$$

we obtain

$$E(x,z) \cong \left[1 + \frac{i}{2k}x(i\phi'' - \phi'^2)\right]\, e^{i\phi} \ , \tag{8.7}$$

where the prime denotes derivative, $\phi' = d\phi/dz$ etc., and we have neglected all terms $\emptyset\left(\frac{\partial^2 E(x,z)}{\partial x^2}\right)$ because we assume that the paraxial approximation holds. Therefore we obtain

$$I(x,z) \cong 1 - \frac{x}{k}\phi'' + \frac{x^2}{4k^2}(\phi''^2 + \phi'^4) \tag{8.8}$$

neglecting higher powers of x. This describes the initial mechanism for the build-up of amplitude fluctuations across the wavefront.

However, we can form *evolution equations*, i.e. differential equations governing the behaviour of the moments. These can be solved to find the far-field. The first few moment equations are trivial in the case of propagation beyond a layer, but are a useful introduction to the moment equations, and illustrate simply some important concepts.

Evolution of the first moment (mean field):
We note first that, since $E(x,z) = 1$ before the screen and $E(0,z) = \exp(i\phi(z))$ immediately after the screen, the initial intensity is unchanged: $< I(0,z) > \equiv 1$, and the initial mean field is

$$m_1(0) = < e^{i\phi(z)} > = e^{-\sigma^2/2} \ . \tag{8.9}$$

This is exact for the normal distribution as assumed here, in which case the the probability density function of ϕ is

$$f(\phi) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\phi^2/2\sigma^2} \ .$$

and approximate in general. It can be obtained from the definition

$$< e^{i\phi} > = \int_{-\infty}^{\infty} e^{i\phi} f(\phi) d\phi \ , \tag{8.10}$$

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or simply by expanding the exponential and averaging term by term,

$$< e^{i\phi} >= 1 + i < \phi > - < \phi^2 > /2 - i < \phi^3 > /3! + ... \tag{8.11}$$

By equation (8.9) we can write the Fourier Transform

$$< \hat{E}(0, \nu) > = \int_{-\infty}^{\infty} m_1(0)\ e^{i\nu z}\ dz\ =\ \sqrt{2\pi}\ \delta(\nu)\ e^{-\sigma^2/2}\ . \tag{8.12}$$

But $\hat{E}$ satisfies the parabolic equation, so

$$\hat{E}(x, \nu) = e^{-i\frac{\nu^2}{2k}x} \hat{E}(0, \nu) \tag{8.13}$$

Taking the average of (8.13) and using (8.12) then gives

$$< \hat{E}(x, \nu) > =\ \sqrt{2\pi}\ \delta(\nu)\ e^{-i\nu^2 x/2k}\ e^{-\sigma^2/2},$$

so that (because of the delta function) $< \hat{E}(x, \nu) >=< \hat{E}(0, \nu) >$ for all z, i.e.

$$\frac{dm_1}{dx}\ =\ 0. \tag{8.14}$$

The mean field is unchanged with distance.

Evolution of the second moment (vertical correlation of field):
We are also interested in mean intensity $< I(x) >$. Although we cannot form an evolution equation for $< I(x) >$ itself, we can do so for $m_2(x, \eta)$ (equation (8.5)) and obtain $< I >$ by solving and setting $\eta = 0$.
The *initial condition* for m_2 at $x = 0$, just beyond the screen, is given by

$$m_2(0, \eta) = \left\langle e^{i[\phi(z_1) - \phi(z_2)]} \right\rangle$$

where $\eta = z_1 - z_2$. Since ϕ is normally distributed, so is the difference $\phi(z_1) - \phi(z_2)$. The variance of this difference is

$$\left\langle [\phi(z_1) - \phi(z_2)]^2 \right\rangle = 2\left[\sigma^2 - \rho(\eta)\right]\ ,$$

where we have denoted by $\rho(\eta)$ the transverse autocorrelation of the layer ϕ:

$$\rho(\eta) =< \phi(z)\phi(z + \eta) > \tag{8.15}$$

with variance

$$\sigma^2 = \rho(0)\ . \tag{8.16}$$

This gives the initial value

$$m_2(0, \eta)\ =\ e^{-[\sigma^2 - \rho(\eta)]}. \tag{8.17}$$

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Now consider the 'transform' moment M_2 defined by

$$M_2(x,\nu_1,\nu_2) = \left\langle \hat{E}(x,\nu_1)\hat{E}^*(x,\nu_2)\right\rangle .$$

By using the parabolic equation (8.6) for the transform field we get

$$\frac{\partial M_2}{\partial x} = \frac{i}{2k}(\nu_2^2 - \nu_1^2)\ M_2 \tag{8.18}$$

However we can write M_2 directly in terms of E, as

$$\begin{aligned} M_2(x,\nu_1,\nu_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle E(x,z_1)E^*(x,z_2)\rangle\ e^{i(\nu_1 z_1-\nu_2 z_2)}\ dz_1\ dz_2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m_2(x,\eta)\ e^{i(\nu_1-\nu_2)Y/2+i(\nu_1+\nu_2)\eta/2}\ d\eta\ dY \end{aligned} \tag{8.19}$$

where we have made the change of variables $\eta = z_1 - z_2$, $Y = z_1 + z_2$. Evaluating the Y-integral in (8.19) gives

$$M_2(x,\nu_1,\nu_2) = \sqrt{2\pi}\ \delta(\nu_1-\nu_2) \int_{-\infty}^{\infty} m_2(x,\eta)\ e^{i(\nu_1+\nu_2)\eta/2}\ d\eta \tag{8.20}$$

so that M_2 vanishes unless $\nu_1 = \nu_2$. Hence we see from equation (8.18) that M_2, and therefore m_2, does not evolve with x, i.e.

$$\frac{\partial M_2}{\partial x} = \frac{\partial m_2}{\partial x} = 0. \tag{8.21}$$

In particular the mean intensity remains constant. (It will be seen later that this no longer holds for an extended random medium.) We therefore need to go to higher moments to describe the intensity fluctuations which the eye and most 'square law' detectors observe in waves propagating through an irregular layer. Before doing that, we shall consider the evolution of the first and second moments in an extended random medium.

8.2 Propagation in an extended random medium

Consider now the second mechanism which can produce field fluctuations, that of extended refractive index irregularities. This is common in many situations, e.g. underwater acoustic, or atmospheric radio wave propagation.

(Apart from any random irregularities there is often an underlying profile; for example the ocean sound channel which causes upward refraction of ray paths, confining sound to a region near the surface. This will not be treated here.)

Consider again a 2-dimensional medium (x, z) and a time-harmonic wave $\phi e^{i\omega t}$. Let $c(x, z)$ be the wave speed in the medium, and c_0 be the 'reference' or average wave speed. (We will take this as constant here although the actual profile may depend on depth.) Let $k = \omega/c_0$ be the corresponding wavenumber.
Denote the **refractive index** by $n(x, z) = c_0/c(x, z)$. We can write

$$n = 1 + n_d(z) + \mu W(x, z) \tag{8.22}$$

where n_d is the deterministic profile which, for example, allows for channelling, but which will be set to zero in the following derivation. μW is the random part, where W has been normalised, so that

$$< W >= 0, \quad < W^2 >= 1,$$

and therefore μ^2 is the variance of n. We will take W to be normally distributed, and *stationary in x and z*. We can then define the 2-dimensional autocorrelation function

$$\rho(\xi, \eta) = \mu^2 \langle W(x, z)\; W(x + \xi, z + \eta)\rangle \tag{8.23}$$

so that $\rho(0, 0) = \mu^2$. **Note** that ρ is assumed to decay to zero as $\xi \to \infty$ or $\eta \to \infty$. (This is reasonable unless there is an underlying periodicity in the medium.)
Further define the *horizontal and vertical length scales H, L* defined by

$$\rho(H, 0) = \rho(0, L) = \mu^2 e^{-1}.$$

There are thus at least three measures affecting the scattering in different ways: μ^2, H, and L. We will look at their various effects on the field.

Weak scatter assumptions: We make the following assumptions, which correspond to different forms of weak scattering restrictions.

(1) *Small variation of refractive index*, $\mu^2 \ll 1$ (or equivalently $|n^2 - 1| \ll 1$).

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(2) *Small angles of scatter*, expressed as

$$\lambda_0 \ll L$$

where λ_0 is the reference wavelength, $\lambda_0 = 2\pi/k_0$.

(3) *Weakly scattering medium*, i.e. the phase fluctuations imposed over a distance H are small,

$$k_0 \mu H \ll 1$$

Note: It can be seen that 'stretching' the scale size H increases the scattering effect, whereas stretching the vertical scale L weakens it. We illustrate this using a heuristic analysis where we shall *ignore diffraction* and examine only the scattering term in the parabolic equation for the wavefield (8.29).
Consider therefore a vertical layer consisting of the region $[x, x{+}d]$. Subdivide this into n thin subregions each of width $\Delta x = d/n$.

Each of these subregions, for $j = 1, ..., n$, imposes a normally-distributed phase change $\phi_j(z)$ with mean zero, whose variance is assumed to be given, say:

$$\langle \phi_j \rangle = 0, \ \left\langle \phi_j^2(z) \right\rangle = \delta^2. \tag{8.24}$$

So since we are ignoring diffraction the wave emerging at $x + d$ has the form

$$E(x+d, z) \ = \ E(x, z) \ e^{i\phi(z)} \tag{8.25}$$

where

$$\phi(z) \ = \ \sum_{i=1}^{n} \ \phi_i(z).$$

Now, since ϕ is normally distributed, the mean of this phase modification is

$$\left\langle e^{i\phi} \right\rangle \ = \ e^{-<\phi^2>/2} \tag{8.26}$$

so we want to examine the dependence of $< \phi^2 >$ on H. Consider two extreme cases:

(1) H small, say $H \leq \Delta x$: Then we can treat ϕ_i, ϕ_j as independent for all $i \neq j$, so that

$$\begin{aligned} \langle \phi^2 \rangle &= \left\langle \left(\sum_{i=1}^{n} \phi_i(z) \right)^2 \right\rangle \\ &= \sum_{i=1}^{n} \langle \phi_i^2(z) \rangle \\ &= n\delta^2 \end{aligned} \tag{8.27}$$

so that scattering scales linearly with n

(2) H large, say $H \gg d$: Then we can suppose that the medium at each depth z is approximately constant over the interval $[x, x + d]$,

$$\phi_i(z) = \phi_j(z) \text{ for all } i, j,$$

so that

$$< \phi^2 > = \langle [n\phi_1(z)]^2 \rangle = n^2 \, \delta^2. \tag{8.28}$$

Thus, increasing H magnifies the scattering effect of the medium.

Moment equations for an extended random medium

We now go on to formulating and solving equations for the evolution of the moments, analogous to those for the thin layer.

Under the weak scatter assumptions we will be able to use the parabolic equation for an extended random medium, which was derived in section 3:

$$\frac{\partial E}{\partial x} = \frac{i}{2k} \frac{\partial^2 E}{\partial z^2} + \frac{ik}{2}(n^2(x, z) - 1)E. \tag{8.29}$$

In the derivations that follow, we shall consider separately the 'scattering effect' and the 'diffraction effect', i.e. the two terms on the r.h.s. of the parabolic equation (8.29).

Define again the **first moment**

$$m_1(x) = < E(x, z) > \tag{8.30}$$

where this quantity is again independent of z by the stationarity of W. Thus all z-derivatives $d^n m_1/dz^n$ vanish, so that *all effects on the mean field are due to the scattering term only* (in eq. (8.29)).

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In order to derive the first moment equation, consider first the phase change $\phi(z)$ over a distance $d > H$ due to the scattering term only:

$$E(x+d,z) = E(x,z)\ e^{i\phi(z)} \tag{8.31}$$

where

$$\phi(z) = k_0\mu \int_x^{x+d} W(x',z)\ dx'. \tag{8.32}$$

Averaging (8.31) will involve a term $< \phi^2 >$, so square and average (8.32) to get

$$\begin{aligned} < \phi^2 > &= k_0^2\mu^2 \int_x^{x+d} \int_x^{x+d} \langle W(x',z)W(x'',z)\rangle\ dx'\ dx'' \\ &= k_0^2\mu^2 \int_x^{x+d} \int_x^{x+d} \rho(x'-x'',0)\ dx'\ dx''\ , \end{aligned}$$

where we have used the definition (8.23) for the transverse autocorrelation $\rho(x'-x'',0)$. We now make the change of variables

$$\begin{aligned} \xi &= x'-x'' \\ X &= (x'+x'')/2 \end{aligned}$$

and use $d > H$ together with the fact that $\rho(\xi,0) \sim 0$ for large ξ to obtain

$$\left\langle \phi^2 \right\rangle \ \cong\ k^2\mu^2 \int_0^d \int_{-\infty}^{\infty} \rho(\xi,0)\ d\xi\ dX.$$

Therefore

$$\left\langle \phi^2 \right\rangle \ =\ k^2\mu^2\sigma_0 d \tag{8.33}$$

where

$$\sigma_0 = \int_{-\infty}^{\infty} \rho(\xi,0)\ d\xi.$$

Now, averaging (8.31) and using (8.33) gives

$$m_1(x+d) \equiv \langle E(x+d,z)\rangle \ \cong\ m_1(x)\ e^{-k_0^2\mu^2\sigma_0 d/2} \tag{8.34}$$

where we have made a further key assumption: *the field becomes independent of the medium*, due to the cumulative effect of scattering., i.e. for large x

$$\left\langle E(x,z)e^{i\phi(z)}\right\rangle \sim \langle E(x,z)\rangle\ \left\langle e^{i\phi(z)}\right\rangle .$$

This is because the moments of E at any distance in the medium are the result of the cumulative effect of the medium (i.e. of scattering) up to that distance, however the statistics of the medium in a layer of thickness d are independent of the statistics of the medium elsewhere.
It now follows directly from (8.34) that

$$m_1(x) \;=\; e^{(-k_0^2\mu^2\sigma_0/2)x}\; m_1(0). \tag{8.35}$$

Equivalently (or expanding $m_1(x+\xi)$ in ξ and comparing terms of $O(\xi)$ with a Taylor series) we can write

$$\frac{dm_1}{dx} \;=\; -(\frac{1}{2}k_0^2\mu^2\sigma_0)m_1\ . \tag{8.36}$$

Thus $m_1(x)$ decays exponentially and is purely real.

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Equation (8.36) for the evolution of the first moment due to the scattering term only, is also valid for a more general incident wave in 3 dimensions, and amplitude different from 1, where the wave emerging from from a screen of width d is given by:

$$E(x+\xi, y, z) = E(x, y, z)e^{\phi(x+\xi,y,z)} \ ,$$

and we have

$$\frac{\partial m_1}{\partial x} \ = \ -(\frac{1}{2}k_0^2\mu^2\sigma_0)m_1 \ . \qquad (8.37)$$

Here again the 'diffraction' term does not appear in the equation, because all higher derivatives in y are also zero by stationarity, and therefore m_1 is independent of the transverse directions y, z.

For higher moments, we shall see that the ∇^2 term must be retained and we need to use

$$\frac{\partial E}{\partial x} = \frac{i}{2k_0}\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right)E + \frac{ik_0}{2}(n^2-1)E \ . \qquad (8.38)$$

The evolution equations could be solved by applying some small perturbations method, for example Born or Rytov, but only for small intensity fluctuations. Such solutions are of very limited use, since we know from experimental results and observations that even small randomness can give rise to very large intensity fluctuations.
It is possible to find a solution that allows for large intensity fluctuations by a *local* application of the method of small perturbation, and we can derive moment equations and their solutions in this way. Conceptually then, using these moment equations to describe the evolution of the field is equivalent to using repeated applications of the Born approximation for successive (thin) screens. In deriving solutions for the moments of the field, we shall treat the scattering and diffraction term separately in the appropriate moment equation, which can be justified by writing the moment equation formally in operator form:

$$\frac{\partial m_i}{\partial x} = [A(x) + B(x)]m_i \ , \qquad (8.39)$$

with formal solution over the range $(x, x+d)$ given by

$$m_i(x+d) = e^{\int_x^{x+d}(A+B)dx'}m_i(x) \ . \qquad (8.40)$$

In the case of a plane wave, $(A+B)$ does not vary with X, so

$$\begin{aligned} m_i(x+d) \ &= \ e^{d(A+B)}m_i(x) \qquad (8.41) \\ &= \ \left(1 + d(A+B) + \frac{d^2}{2}(A+B)^2 + ...\right)m_i(x) \ . \end{aligned}$$

Let us now consider the **second moment**

$$m_2 =< E_1(x, y_1, z_1)E_2^*(x, y_2, z_2) > , \tag{8.42}$$

where E_1 and E_2 represent E at two separate points in the same transverse plane at x.
Let us derive first the 'diffraction' term (or 'distance effect'). Consider

$$\frac{\partial}{\partial x}E_1E_2^* = E_2^*\frac{\partial E_1}{\partial x} + E_1^*\frac{\partial E_2}{\partial x} .$$

The diffraction term for the field at a single point E_j (where $j = 1, 2$) is

$$\frac{\partial E_j}{\partial x} = \frac{i}{2k_0}\left(\frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2}\right)E_j \equiv -\frac{i}{2k_0}\nabla_{Tj}^2 E_j . \tag{8.43}$$

Therefore

$$\frac{\partial}{\partial x}E_1E_2^* = \frac{i}{2k_0}\left(E_2^*\nabla_{T1}^2E_1 - E_1\nabla_{T2}^2E_2^*\right) ,$$

and taking the ensemble average

$$\frac{\partial}{\partial x} < E_1E_2^* >= \frac{i}{2k_0}\left(\nabla_{T1}^2 - \nabla_{T2}^2\right) < E_1E_2^* > . \tag{8.44}$$

We shall now consider the 'scattering' effect due to a screen of thickness d, so how the second moment $< E_1E_2^*(x) >$ evolves onto $< E_1E_2^*(x + d) >$. We have:

$$E_1E_2^*(x + d) = E_1E_2^*(x)e^{i[\phi(x+d,y_1,z_1)-\phi(x+d,y_2,z_2)]} \tag{8.45}$$

and

$$E_1E_2^*(x + d) \simeq E_1E_2^*(x) + \frac{\partial}{\partial x}(E_1E_2^*)d , \tag{8.46}$$

so

$$E_1E_2^*(x+d) = E_1E_2^*(x)[1+i(\phi(y_1, z_1)-\phi(y_2, z_2))-\frac{1}{2}(\phi(y_1, z_1)-\phi(y_2, z_2))^2+...] , \tag{8.47}$$

where we have expanded the exponent in (8.45). If we now equate (8.45) and (8.46), by taking the ensemble average, and remembering that $< \phi >= 0$, we have

$$\frac{\partial}{\partial x} < E_1E_2^* > d = -\frac{1}{2}\left\langle(\phi(y_1, z_1) - \phi(y_2, z_2))^2\right\rangle < E_1E_2^* > . \tag{8.48}$$

Now consider

$$\left\langle(\phi_1 - \phi_2)^2\right\rangle = \left(< \phi_1^2 > -2 < \phi_1\phi_2 > + < \phi_2^2 >\right) , \tag{8.49}$$

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where $\phi_i = \phi(y_i, z_i)$ In the same way as we previously derived $< \phi^2 >$, (equation (8.33)), we can derive $\langle\phi_1\phi_2\rangle$:

$$\langle\phi_1\phi_2\rangle \cong k_0^2\mu^2 d\sigma(\eta,\zeta) \tag{8.50}$$

where

$$\sigma(\eta,\zeta) = \int_{-\infty}^{\infty} \rho(\xi,\eta,\zeta)\ d\xi,$$

$\eta = y_1 - y_2$, $\zeta = z_1 - z_2$, and ρ is the normalised autocorrelation function of the refractive index fluctuation:

$$\rho(\xi,\eta,\zeta) =< W(x_1,y_1,z_1)W(x_2,y_2,z_2) > \ .$$

We can now use (8.50) in (8.49) to obtain

$$\left\langle(\phi_1-\phi_2)^2\right\rangle = 2k_0^2\mu^2 d(\sigma(0,0)-\sigma(\eta,\zeta)) \ . \tag{8.51}$$

Therefore the evolution due to the scattering only is

$$\frac{\partial}{\partial x} < E_1E_2^* >= -k_0^2\mu^2(\sigma(0,0)-\sigma(\eta,\zeta)) < E_1E_2^* > \tag{8.52}$$

Now, combining (8.52) and (8.44), we obtain the second moment equation

$$\frac{\partial m_2}{\partial x} = \frac{i}{2k_0}\left(\nabla_{T1}^2 - \nabla_{T2}^2\right) m_2 - k_0^2\mu^2(\sigma(0,0)-\sigma(\eta,\zeta))m_2 \ . \tag{8.53}$$

Solution of the second moment equation *(non-examinable)*
It is convenient to use the set of variables

$$\begin{aligned} \xi = x_1 - x_2 \ , \ \eta = y_1 - y_2 \quad , \quad \zeta = z_1 - z_2 \\ X = x_1 + x_2 \ , \ Y = y_1 + y_2 \quad , \quad Z = z_1 + z_2 \end{aligned} \tag{8.54}$$

and to set

$$< E_1E_2^* > e^{\beta x} = u(\beta x, \eta, \zeta) \ . \tag{8.55}$$

The equation for the second moment then can be written as

$$\frac{\partial u}{\partial(\beta x)} = -\frac{2i}{k_0\beta}\left(\frac{\partial^2 u}{\partial Y\partial\eta} + \frac{\partial^2 u}{\partial Z\partial\zeta}\right) + \frac{\sigma(\eta,\zeta)}{\sigma(0,0)}u \ . \tag{8.56}$$

It is convenient to transform this equation using the transform pair

$$\begin{aligned} u(Y,\eta;Z,\zeta) &= \int\int \hat{u}(\eta,\zeta;\epsilon_1,\epsilon_2)e^{i(\epsilon_1 Y+\epsilon_2 Z)}d\epsilon_1 d\epsilon_2 \ , \\ \hat{u}(\eta,\zeta;\epsilon_1,\epsilon_2) &= \frac{1}{2\pi}\int\int u(Y,\eta;Z,\zeta)e^{-i(\epsilon_1 Y+\epsilon_2 Z)}dY dZ \ . \end{aligned}$$

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to obtain

$$\frac{\partial \hat{u}}{\partial(\beta x)} = B_1 \frac{\partial \hat{u}}{\partial \eta} + C_1 \frac{\partial \hat{u}}{\partial \zeta} + \frac{\sigma(\eta,\zeta)}{\sigma(0,0)} \hat{u} \ , \tag{8.57}$$

where

$$B_1 = 2\epsilon_1/k\beta \ , \ C_1 = 2\epsilon_2/k\beta \ .$$

The general solution of (8.57)is

$$\hat{u} = \hat{u}_0(\eta + B_1\beta x, \zeta + C_1\beta x) \exp\left[\int_0^{\beta x} \frac{\sigma(\eta + B_1(\beta x - \beta x'); \ \zeta + C_1(\beta x - \beta x'))}{\sigma(0,0)} d(\beta x')\right] \ , \tag{8.58}$$

where $\hat{u}_0$ is the solution of the transform equation (8.57) when $\sigma(\xi,\eta,\zeta) = 0$. The second moment then is given by the inverse transform, which, in our case where $Y = Z = 0$, reduces to

$$u(Y,\eta; Z,\zeta) = \int\int \hat{u} d\epsilon_1 d\epsilon_2 \ . \tag{8.59}$$

If the incident field is a plane wave with amplitude E_0 at $x = 0$ and propagating parallel to the x-direction, then

$$\hat{u}_0 = \delta(\epsilon_1)\delta(\epsilon_2) \tag{8.60}$$

and from (8.58) and the inverse transform we have:

$$\begin{aligned} u &= E_0^2 \exp\left[\int_0^{\beta x} \frac{\sigma(\eta;\zeta) d(\beta x')}{\sigma(0,0)}\right] \\ &= E_0^2 \exp\left[-\beta x \left(1 - \frac{\sigma(\eta;\zeta)}{\sigma(0,0)}\right)\right] \ . \end{aligned} \tag{8.61}$$

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Part III
The inverse scattering problem

9 Introduction to inverse problems

9.1 Ill-posedness

We have so far only considered the *direct* scattering problem, i.e., given a wavefield u_i incident upon an inhomogeneity (this could be an interface such as an infinite surface or a finite, closed object, or an extended inhomogeneity such as a medium with varying refractive index), we have considered ways of finding the scattered field u_s, or equivalently the total field $u = u_i + u_s$.
The *inverse* scattering problem starts from the knowledge of the scattered field u_s, and asks questions about the inhomogeneities that produced it (for example their shape, or their refractive index) or about the source field.
This area of research is fairly new, because the nature of the problem gives rise to a mathematical problem which is *ill-posed*, and until about the '60's was not considered worth studying from a mathematical point of view.

The usual definition of a *'well-posed'* problem is due to Hadamard: a problem is **well-posed** if

1. There exists a solution to the problem (**existence**)
2. There is at most one solution (**uniqueness**)
3. The solution depends continuously on the data (**stability**)

We shall usually express an inverse problem formally as: given y, find x s.t.

$$Ax = y \ , \tag{9.1}$$

where A is an operator from a normed space X into a normed space Y, $A : X \mapsto Y$. Only linear operators will be considered throughout, unless explicitly stated.
The requirements for well-posedness listed above translate into the following properties of the operator A:

1. A **is surjective**, i.e. $\forall y \in Y \ \exists x \in X$ s.t. $Ax = y$. If A is not surjective, then equation (9.1) is not solvable for all $y \in Y$ (*non-existence*).

2. A **is injective**, i.e. $\forall x_i, x_j \in X, x_i \neq x_j \implies Ax_i \neq Ax_j$. If A is not injective, then equation (9.1) may have more than one solution (*non-uniqueness*)

3. A^{-1} **is continuous**, i.e. $\forall$ sequences $x_n \in X$ with $Ax_n \to Ax$ as $n \to \infty$, it follows that $x_n \to x$ as $n \to \infty$. If this is not the case, then there

may be cases when for $\| y' - y \| \ll 1$ we have $\| x' - x \| \gg 1$: small differences in y (e.g. small errors in the measurement or in the numerical computation) give rise to large errors in the solution (*instability*).
We note that a linear operator is continuous iff it is **bounded**.
Recall: a linear operator $A : X \mapsto Y$ is called bounded if $\exists$ a constant C such that $\forall x \in X$ we have $\| Ax \| \leq C \| x \|$.
Absence of even one of these three properties is likely to pose considerable difficulties in finding the solution. It turns out that most inverse problems (i.e. 'given y find x s.t. $Ax = y$', or equivalently $x = A^{-1}y$) are ill-posed, whilst their corresponding direct problems (i.e. given x 'find y s.t. $Ax = y$', or equivalently $y = Ax$) are well-posed.
We shall see how it is is possible to deal with the difficulties caused by ill-posedness by constructing suitable approximations to the inverse operator A^{-1}.

- Lack of of properties (1) and (2) can be counteracted by using the ***'Moore-Penrose generalised inverse'*** (usually denoted by $A^{\dagger}$).
- Lack of property (3) can be counteracted by using a suitable ***regularisation***.

We shall now consider a simple concrete example of an inverse problem, which is very useful to illustrate some effects that are typical for ill-posed problems. Integration and differentiation are inverse to each other. It is quite arbitrary to decide which one to consider as the direct problem, but we shall choose to consider differentiation as the inverse problem, since it presents some of the difficulties associated with ill-posedness.
Let us therefore consider functions $f(x)$ continuous on the interval $[0, 1]$, and consider our ***direct problem*** to be the evaluation of the integral

$$Af(x) = \int_0^x f(x')dx' \tag{9.2}$$

given f.
The ***inverse problem*** is then solving

$$Af(x) = g(x) \tag{9.3}$$

for f, given g, so $f(x) = A^{-1}g(x) = g'(x)$.
The given data will in general be noisy (measured with some error), so we shall actually have a $g^{(\delta)}$, with $\| g^{(\delta)} - g \| \leq \delta$. Take, for example, the functions g_n^{δ} defined by:

$$g_n^{(\delta)}(x) = g(x) + \delta \sin \frac{nx}{\delta} \ , \ n \in \mathbb{N} \tag{9.4}$$

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which satisfy $\| g_n^{(\delta)} - g \| = \delta \ \forall n$.

The inverse gives

$$g_n^{(\delta)\prime}(x) = g'(x) + n \cos \frac{nx}{\delta} \ , \tag{9.5}$$

and we have $\| g_n^{(\delta)\prime} - g' \| = n$, which can be arbitrarily large. Therofore, the solution of the inverse probem does not depend continuously on the date.
It is interesting to consider differention as our inverse problem, also because it provides a very typical example of how noisy data affect the total error in the solution of the inverse problem.
We can compute an approximate solution of (9.5) by using central differences with stepsize h:

$$\frac{g(x+h) - g(x-h)}{2h} = g'(x) + O(h) \ . \tag{9.6}$$

But we have noisy data $g^{(\delta)}$, so we actually compute:

$$\begin{aligned} \frac{g^{(\delta)}(x+h) - g^{(\delta)}(x-h)}{2h} & \simeq \\ \frac{g(x+h) - g(x-h)}{2h} + \frac{\delta}{h} & = g'(x) + O(h) + \frac{\delta}{h} \end{aligned} \tag{9.7}$$

Therefore the total error in our inverse problem with noisy data is the sum of an approximation error (which tends to 0 as $h \to 0$), and an error due to the noisy data (which tends to infinity as $h \to 0$).
We can also see that we can use a parameter, the step-size in this case, to restore stability. If we plot $\| g_n^{(\delta)\prime} - g' \|$ against h, the minimum of the curve gives us the optimal value of the parameter, h_0, which of course depends on δ. This is very typical of inverse problems.

9.2 The Moore-Penrose generalised inverse

Let us now return to the problem of approximating the inverse A^{-1} in order to obtain a well-posed problem. We shall first address the difficulty caused by non-existence.

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The **generalised inverse** $x^\dagger$ is a 'generalised solution' to $Ax = y$ defined as follows:

$x^\dagger$ is the element with smallest norm in the set of minimisers of $\| Ax - y \|$.
It can be shown that $x^\dagger$ is the unique solution to

$$A^* A x = A^* y \ . \tag{9.8}$$

The linear operator $A^\dagger$ defined by $x^\dagger = A^\dagger y$ is called the **Moore-Penrose generalised inverse**.
The following applies:

- The Moore-Penrose generalised inverse always exists and is unique.
- If A^{-1} exists, then $A^\dagger = A^{-1}$.
- $Ker(A^\dagger) = Ker(A^*)$ and $\mathcal{R}(A^\dagger) = \mathcal{R}(A^*)$.
- $\mathcal{D}(A^\dagger) = \mathcal{R}(A) + \mathcal{R}(A)^\perp$.

Here $\mathcal{R}(K)$ denotes the range of an operator K, $\mathcal{D}(K)$ denotes its domain, and $Ker(K)$ denotes the kernel of an operator K, i.e. the set of of elements in $\mathcal{D}(K)$ which are mapped to zero. $\mathcal{R}(A)^\perp$ is the orthogonal complement of $\mathcal{R}(A)$, i.e. the set of all elements in X which are orthogonal to $\mathcal{R}(A)$.
We note here that, in order to define orthogonality, we need to have an *inner product*, and two elements x and y are said to be orthogonal if their inner product (x, y) is zero.

> [*Recall*: an inner product of two elements x and y of a vector space X over a field $\mathbb{F}$ is a function with the following properties:
> 1. $(x, y) \geq 0$ and $(x, x) = 0 \Leftrightarrow x = 0$,
> 2. $(x, y) = \overline{(y, x)}$, where $\overline{(\cdot, \cdot)}$ denotes the complex conjugate
> 3. $(ax, y) = a(x, y)\ \forall a \in \mathbb{F}$ (and $(x, ay) = a^*(x, y)$)
> 4. $(x + y, z) = (x, z) + (y, z)$.]

Given an inner product, it is always possible to obtain a norm induced by the inner product putting $\| x \|^2 = (x, x)$. However, not all norms induce an inner product. If the norm defined on X satisfies

$$2 \| x \|^2 + 2 \| y \|^2 = \| x + y \|^2 + \| x - y \|^2 \ , \tag{9.9}$$

then the norm induces an inner product (x, y) s.t. $\| x \|^2 = (x, x)$. For example, for vectors with the Euclidean norm the inner product is the dot

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product; for square integrable functions f and g in L^2 with the l_2 norm the inner product is:

$$(f, g) = \int_{-\infty}^{+\infty} fg \ .$$

In finite-dimensional spaces, where the operator A can be written as a matrix, it is straightforward to show that, if there exists a singular value decomposition (SVD) for A: $A = U\Sigma V^*$, then $A^\dagger = V\Sigma^* U^*$.

> [*Recall (linear algebra)*: if A is a $k \times n$ matrix, then SVD is a factorisation of A: $A = U\Sigma V^*$ (V^* denoting the conjugate transpose of V) where U is a $k \times k$ unitary matrix (i.e. $UU^* = U^*U = \mathbb{I}$); V^* is a $n \times n$ unitary matrix Σ is a diagonal $k \times n$ matrix with non-negative real diagonal elements.]

In infinite-dimensional spaces, if an operator A is compact (a stronger property than just bounded, and here we shall just note that all integral operators with square-integrable kernel are compact) we can use a generalisation of the SVD to construct the generalised inverse: the **singular value system**.
A singular value system for A is the set $\{\sigma_i; u_i; v_i\}$, $i \in \mathbb{N}$, where:

- σ_i^2 are the eigenvalues of the self-adjoint operator A^*A (and therefore also of AA^*);
- v_i are the corresponding complete orthonormal system of eigenvectors of A^*A;
- u_i are defined by $u_i = \frac{Av_i}{\|Av_i\|}$, so are a complete orthonormal system of eigenvectors of AA^*;

Given a singular value system, we can write

$$\begin{aligned} Av_i &= \sigma_i u_i & \qquad A^*Av_i &= \sigma_i^2 v_i \\ A^*u_i &= \sigma_i v_i & \qquad AA^*u_i &= \sigma_i^2 u_i \end{aligned}$$

and we have, $\forall x \in X$ and $y \in Y$, the following singular value expansions:

$$\begin{aligned} Ax &= \sum_{i=1}^{\infty} \sigma_i (x, v_i) u_i \\ A^*y &= \sum_{i=1}^{\infty} \sigma_i (y, u_i) v_i \end{aligned}$$

The Moore-Penrose generalised inverse can then be costructed by:

$$A^{\dagger}y = \sum_{i=1}^{\infty} \frac{(y, u_i)}{\sigma_i} v_i \ . \tag{9.10}$$

We can see immediately from this expression that $A^{\dagger}$ is not in general bounded.

It is possible, though, even when A^{-1} is not bounded, but has dense range, to construct a family of bounded approximation to A^{-1}. A strategy for achieving this is the Tikhonov regularisation procedure, which provides a mean to cope with ill-posedness.

9.3 Tikhonov regularisation

Definition A regularisation strategy for $A : X \mapsto Y$ is a family of bounded linear operators $R_\alpha : Y \to X$ for $\alpha > 0$ such that

$$R_\alpha y \to A^{-1}y \text{ as } \alpha \to 0 \tag{9.11}$$

To do this, we note that, if a solution to the inverse scattering problem $Ax = y$ exists, it minimises

$$\| Ax - y \| = 0 \ . \tag{9.12}$$

So far, we have achieved existence and uniqueness by introducing the Moore-Penrose generalised inverse and seeking as our solution the element x with the smallest norm that minimises (9.12), which is the unique solution to the normal equation

$$A^* A x' = A^* y \ , \tag{9.13}$$

(where $A^* : Y \mapsto X$).
Equation (9.13) is still ill-posed, if the original scattering problem was ill-posed, but this ill-posedness can be removed by introducing a small perturbation, so replacing the original problem with the slightly perturbed one below:

$$\alpha x_\alpha + A^* A x_\alpha = A^* y \tag{9.14}$$

for some small $\alpha > 0$.
It is possible to prove that (**Theorem**):
If $\alpha > 0$, then the operator $(\alpha I + A^* A) : X \mapsto X$ has an inverse, which is bounded, with $\| (\alpha I + A^* A)^{-1} \| \leq \alpha^{-1}$.

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9.3 Tikhonov regularisation

Given a linear bounded operator $A : X \mapsto Y$, and $y \in Y$, the ***Tikhonov functional*** is defined by

$$J_\alpha = \| Ax - y \|^2 + \alpha \| x \|^2 \quad \forall x \in X \tag{9.15}$$

For $\alpha > 0$, the Tikhonov functional J_α, as defined above, has a unique minimum x_α given as the unique solution of the equation

$$\alpha x_\alpha + A^* A x_\alpha = A^* y \ . \tag{9.16}$$

The solution of this equation can be written as $x_\alpha = R_\alpha y$, with

$$R_\alpha = (\alpha I + A^* A)^{-1} A^* \ : Y \mapsto X. \tag{9.17}$$

$x_\alpha = R_\alpha y$ is referred to as the *Tikhonov regularisation solution* of (9.1). This strategy then approximates the actual solution $x = A^{-1} y$ by the regularised solution x_α, given y. In general, a y_δ will be known, which differs from y by some error δ (for example because it is experimental data):

$$\| y_\delta - y \| \leq \delta \ . \tag{9.18}$$

It is useful to be able to approximate the error involved in the regularisation, and to relate it to the error associated with incorrect initial data δ. Let's write

$$x_{\alpha(\delta)} - x = R_\alpha y_\delta - R_\alpha y + R_\alpha A x - x \ . \tag{9.19}$$

Then, by the triangle inequality we have the estimate

$$\| x_{\alpha(\delta)} - x \| \leq \delta \| R_\alpha \| + \| R_\alpha A x - x \| \tag{9.20}$$

This decomposition shows that again the error consists of two parts: the first term reflects the influence of the incorrect data, and the second term is due to the approximation error between R_α and A^{-1}.
The regularisation scheme requires a strategy for choosing the parameter α on the basis of the error δ in the data, in order to achieve an acceptable total error for the regularised solution.

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10 Solving inverse problems

Remarks

It is worth remembering that there are several types of inverse scattering problems, depending on what property of the scatterer is considered unknown. We shall concentrate first on the problem of reconstructing the geometry of the scatterer, then we shall consider the problem of reconstructing the refractive index, and will only present a few simple results.

Inverse problems have been treated from many points of view.

- Some exact solutions, depending on the geometry of the scatterer, are available. They are usually based on expressing the surface of the scatterer parametrically in a coordinate system in which the Helmholtz equation is separable.

- Some methods exploit the properties of the far field in order to construct an analytical continuation of the far field into the near field of the scatterer, and the circle of minimum radius enclosing the scatterer, then determine enough points on the scatterer to approximate its shape sufficiently. The method of Imbriale and Mittra comes in this category.

  These methods need some a priori knowledge of the boundary condition at the surface of the scatterer, so will not work for many real problems where only very limited knowledge of the scatterer is available. They are very unstable, and also will usually not work in the 'resonance region', where the size of the incident wavelength is comparable to the size of the scatterer.

- Other methods which exploit the properties of the far field use an 'indicator function' to construct the shape of the scatterer on a grid covering some area which is 'searched' for the unknown scatterer. The ***linear sampling method*** and some of its variants are in this category. They are often based on a factorization of the 'far field operator', which maps the incident field onto the far field.

  These methods do not need a priori knowledge of the scatterer (although some idea of the location of the scatterer(s) greatly helps!). They can be used in the resonance region, and can also be used for scatterer which are not simply connected.

  In principle, though, these methods usually need knowledge of the far field in *all* direction, which of course is very often impossible to obtain.

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- A number of methods based on ***iterative*** procedures are also available. They need some a priory knowledge to make an initial guess for the first iterate, which in practice though does not need to be particularly close to the exact solution at all. Their implementation usually means linearising the problem.

  Many of these methods need to solve the forward problem at least once at each iteration step, but there are also other, more efficient, that do not.

  When the data are sufficiently good, and the scatterer or inhomogeneities sufficiently 'smooth', they can provide very high-quality reconstruction.

  These methods are particularly suited to lower frequencies, and for problems of scattering by extended inhomogeneities.

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10.1 Shape reconstruction

The first two methods we are going to present here are based on the properties of the far field. Therefore, before we proceed, we shall first give some definitions related to the far field, and recall some results we have already seen. Recall that, in the direct scattering problem, given an incident (time-harmonic) field ϕ_i with direction defined by the incident wavevector $\mathbf{k} = k\mathbf{y}$ on a (bounded) scatterer V with boundary ∂V, we seek the total field

$$\phi = \phi_i + \phi_s \ ,$$

such that ϕ_s obeys the Helmholtz equation with suitable boundary conditions on ∂V, and the radiation condition at infinity, which can be expressed as

$$r^{1/2}\left(\frac{\partial \phi_s}{\partial r} - ik\phi_s\right) \to 0 \quad \text{as } r \to \infty \quad \text{(in 2D)} \tag{10.1}$$

$$r\left(\frac{\partial \phi_s}{\partial r} - ik\phi_s\right) \to 0 \quad \text{as } r \to \infty \quad \text{(in 3D).} \tag{10.2}$$

Here we shall take the incident field as a plane wave

$$\phi_i(\mathbf{x}, \mathbf{y}, k) = e^{ik\mathbf{x}\cdot\mathbf{y}} \ , \tag{10.3}$$

(readily extended to any superposition of plane waves). Then there exists a function $f_\infty(\hat{\mathbf{x}}, \mathbf{y}, k)$ such that

$$\phi_s(\mathbf{x}, \mathbf{y}, k) = \frac{e^{kr}}{r^{1/2}} \left(f_\infty(\hat{\mathbf{x}}, \mathbf{y}, k) + O\left(\frac{1}{r^{1/2}}\right)\right) \quad \text{(in 2D)} \quad (10.4)$$

$$\phi_s(\mathbf{x}, \mathbf{y}, k) = \frac{e^{kr}}{r} \left(f_\infty(\hat{\mathbf{x}}, \mathbf{y}, k) + O\left(\frac{1}{r}\right)\right) \quad \text{(in 3D)} \quad (10.5)$$

as $r \to \infty$. Here $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $r = |\mathbf{x}|$.
The function f_∞ is the ***far field amplitude***, also called ***directivity pattern***. We shall drop herefter explicit dependence on k, which is the same throughout. It is sometimes useful to expand $f(\hat{\mathbf{x}}, \mathbf{y})$ in terms of spherical harmonics:

$$f(\hat{\mathbf{x}}, \mathbf{y}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} Y_n^m(\hat{\mathbf{x}}) \quad (10.6)$$

It is well known and it can be proven (see, e.g., Colton and Kress 1992), that

Theorem: *exact knowledge of the far field amplitude* f uniquely *determines the scatterer* V.
Note that this tells us that we can use f to determine the scatterer, but it doesn't tell us *how*! nor does it tell us whether we can still reconstruct the scatterer without *exact* knowledge of f, which is always the case in real life, where the far field is known only at a finite, discrete number of points, and with some experimental error and noise.

We shall now define the far field operator and the Herglotz function.
Given a far field amplitude $f_\infty(\hat{\mathbf{x}}, \mathbf{y})$, the **Far Field Operator** $F : L^2(S_1) \to L^2(S_1)$ is defined by:

$$(Fg)(\hat{\mathbf{x}}) = \int_{S_1} f_\infty(\hat{\mathbf{x}}, \mathbf{x_1}) g(\mathbf{x_1}) ds(\mathbf{x_1}) \ , \quad (10.7)$$

where S_1 is the unit sphere, and $g(\mathbf{x})$ is a suitably well-behaved function on S_1 ($g \in L^2(S_1)$).
The **Herglotz wave function** with kernel g is defined by

$$v_g(\mathbf{x}) = \int_{S_1} e^{ik\hat{\mathbf{x}}_1 \cdot \mathbf{x}} g(\hat{\mathbf{x}}_1) ds(\hat{\mathbf{x}}_1) \ , \ x \in \mathbb{R}^3 \ . \quad (10.8)$$

It is effectively a superposition of plane waves with weights $g(\hat{\mathbf{x}}_1)$. Herglotz wave functions satisfy the Helmholtz equation, and functions defined by

$$v(\mathbf{x}) = \int_{S_1} e^{-ik\hat{\mathbf{x}}_1 \cdot \mathbf{x}} g(\hat{\mathbf{x}}_1) ds(\hat{\mathbf{x}}_1) \ , \ \mathbf{x} \in \mathbb{R}^3 \ . \quad (10.9)$$

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are also Herglotz functions.
Corresponding to an incident Herglotz wave function v_i in a scattering problem, we can associate a scattered field v_s and a far field amplitude v_∞ as follows:

$$v_s(\mathbf{x}) = \int_{S_1} \phi_s(\mathbf{x}, \hat{\mathbf{x_1}}) g(\hat{\mathbf{x_1}}) ds(\hat{\mathbf{x_1}}) \ , \ \mathbf{x} \in \mathbb{R}^3 \setminus_V \ , \tag{10.10}$$

and

$$v_\infty(\mathbf{x}) = \int_{S_1} \phi_\infty(\hat{\mathbf{x}}, \hat{\mathbf{x}}_1) g(\hat{\mathbf{x_1}}) ds(\hat{\mathbf{x}}_1) \ . \tag{10.11}$$

Then $(Fg)(\hat{\mathbf{x}})$, as defined in (10.7) is the far field corresponding to an incident field defined by a Herglotz function with kernel g: $v_g(\hat{\mathbf{x}})$, given by (10.8) above.
We should also note here, without proof, the following result, which relates the far field to the solutions of the Dirichlet problem inside the scatterer V: *if there exist a non-trivial Herglotz function that vanishes on the surface of the scatterer, ∂V, then the far-field patterns are not complete.*
This means that for the far field patterns to be complete in $L^2(S_1)$, the eigenvalue k^2 must not coincide with the eigenvalues of the interior Dirichlet problem.

Colton and Monk's optimization method
A method which was first introduced by Colton & Monk (1985) for the scattering from an acoustically soft surface $S = \partial V$, finds the shape of a scatterer as the minimiser of an expression involving the far field and the kernel of a suitably chosen Herglotz function.
Suppose that a time-harmonic plane wave

$$\phi_i = e^{ik\mathbf{y}\cdot\mathbf{x}} \tag{10.12}$$

is incident upon a scattering surface S that encloses the origin. The total potential $\phi = \phi_i + \phi_s$ satisfies the Helmholtz equation outside S and the boundary condition

$$\phi(\mathbf{x}) = 0 \ , \quad \text{for } \mathbf{x} \text{ on } S \ . \tag{10.13}$$

We are given the incident field ϕ_i and the far field ϕ_∞.
We make the following additional assumptions:
1. k^2 is *not* one of the interior eigenvalues of the interior Dirichlet problem (this is necessary for existence and uniqueness of the solution, but in practice the errors in a numerical calculations will be sufficient to ensure that the eigenvalues of the interior Dirichlet problem are avoided);
2. the surface of the scatterer encloses the origin $\mathbf{x} = 0$;

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2. the scatterer is "starlike", i.e. its surface S can be represented in the form

$$\mathbf{x} = r_s(\hat{\mathbf{x}})\hat{\mathbf{x}} \text{ , for } \mathbf{x} \text{ on } S \text{ ,}$$

where r_s is single-valued (i.e. the surface of the scatterer can be parametrised by a single-valued function of angle - this is not strictly necessary, but is convenient for the calculations).

The problem then is to determine the function $r_s(\hat{\mathbf{x}})$ that specifies the scattering surface S.

We know that the scattered field ϕ_s satisfies Sommerfeld radiation condition:

$$r\left(\frac{\partial \psi}{\partial r} - ik\psi\right) \to 0 \text{ as } r \to \infty \text{ ,} \tag{10.14}$$

and there exists a far field pattern $f_\infty(\hat{\mathbf{x}}, \mathbf{y})$ such that

$$\phi_s(\mathbf{x}, \mathbf{y}, k) = \frac{e^{ikr}}{r}\left[f_\infty(\hat{\mathbf{x}}, \mathbf{y}) + O\left(\frac{1}{r}\right)\right] \text{ .} \tag{10.15}$$

From the Kirchhoff-Helmholtz equation, without assuming Dirichlet boundary conditions and now dropping the explicit dependence on $\mathbf{y}$ and k, we have:

$$\phi_s(\mathbf{x}) = \int_S \left[\phi(\mathbf{x}')\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} - \frac{\partial \phi(\mathbf{x}')}{\partial n} G(\mathbf{x}, \mathbf{x}')\right] ds(\mathbf{x}') \text{ ,} \tag{10.16}$$

where n denotes the outward normal from S, and we can derive the following integral equation for the far field:

$$f_\infty(\hat{\mathbf{x}}; \mathbf{y}) = \frac{1}{4\pi}\int_S \left[\phi_s(\mathbf{x}')\frac{\partial e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}'}}{\partial n} - \frac{\partial \phi_s(\mathbf{x}')}{\partial n} e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}'}\right] ds(\mathbf{x}') \text{ .} \tag{10.17}$$

The derivation above comes from using in (10.16) the far field approximation for the Green's function and for its derivative:

$$G(\mathbf{x}, \mathbf{x}') = \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{4\pi|\ \mathbf{x} - \mathbf{x}'\ |} = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}\left[e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}} + O\left(\frac{1}{|\mathbf{x}|}\right)\right] \tag{10.18}$$

$$\frac{\partial G(\mathbf{x}, \mathbf{x}')}{\partial n} = \frac{e^{ik|\mathbf{x}|}}{4\pi|\mathbf{x}|}\left[-ik\mathbf{n}\cdot\hat{\mathbf{x}}e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}} + O\left(\frac{1}{|\mathbf{x}|}\right)\right] \tag{10.19}$$

and comparing with (10.15). Taking now the given Dirichlet boundary condition into account, we have for the total field:

$$\phi(\mathbf{x}) = \phi_i(\mathbf{x}) - \frac{1}{4\pi}\int_S \frac{e^{ikr}}{r}\frac{\partial \phi(\mathbf{x}')}{\partial n} d\mathbf{x}' \text{ ,} \tag{10.20}$$

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and for the far field:

$$f_\infty(\hat{\mathbf{x}}, \mathbf{y}, k) = -\frac{1}{4\pi}\int_S e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}'}\frac{\partial\phi(\mathbf{x}')}{\partial n}d\mathbf{x}' \ . \tag{10.21}$$

Recall that there exists a one-to-one correspondence between radiating solutions of the Helmholtz equation and their far field pattern. Therefore the problem can be solved by finding an expression that, for a fixed k, relates the far field pattern to the shape of the scatterer, $r_s(\hat{\mathbf{x}})$. The method of Colton and Monk provides this relation via the kernel $g(\mathbf{x})$ of a Herglotz function which is chosen to be equal to the solution $\psi(\mathbf{x}, k)$ of the interior Dirichlet problem for the scatterer.
The function $\psi(\mathbf{x}, k)$ satisfies

$$(\nabla^2 + k^2)\psi(\mathbf{x}, k) = 0 \quad \mathbf{x} \text{ inside } S \ , \tag{10.22}$$

with

$$\psi(\mathbf{x}, k) = \frac{e^{ikr_s}}{4\pi r_s} \ , \quad \mathbf{x} \text{ on } S \ , \tag{10.23}$$

i.e. ψ is the scattered potential inside S induced by a point source at the origin, and in (10.23) we have used the notation r_s to emphasize that here, since $\mathbf{x}$ is on the surface of the scatterer, $|\,x| \ = r \equiv r_s$.
Now define S_1 to be the sphere of unit radius and centre at the origin. We shall therefore multiply (10.21) by a suitable function $g(\hat{\mathbf{x}})$ and integrate with respect to $\hat{\mathbf{x}}$ over the unit sphere centred at the origin S_1:

$$\int_{S_1} f_\infty(\hat{\mathbf{x}}, \mathbf{y}, k)g(\hat{\mathbf{x}})d\hat{\mathbf{x}} = -\int_{S_1}\int_S \frac{e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}'}}{4\pi}\frac{\partial\phi(\mathbf{x}')}{\partial n}g(\hat{\mathbf{x}})d\mathbf{x}'d\hat{\mathbf{x}} \ . \tag{10.24}$$

The integrals on the r.h.s. can be switched, to get:

$$-\int_S \left[\int_{S_1} \frac{e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}'}}{4\pi}g(\hat{\mathbf{x}})\right]\frac{\partial\phi(\mathbf{x}')}{\partial n}d\mathbf{x}'d\hat{\mathbf{x}} = -\int_S \psi(\mathbf{x}')\frac{\partial\phi(\mathbf{x}')}{\partial n}d\mathbf{x}' \tag{10.25}$$

where we have identified the expression in square brackets above as a Herglotz function $v(\mathbf{x})$ with kernel $g(\mathbf{x})$ such that $v(\mathbf{x}) = \psi(\mathbf{x}) =$ solution to the interior Dirichlet problem, i.e.

$$\psi(\mathbf{x}') = \int_{S_1} \frac{e^{-ik\hat{\mathbf{x}}\cdot\mathbf{x}'}}{4\pi}g(\hat{\mathbf{x}})d\hat{\mathbf{x}} \tag{10.26}$$

We can now re-write this by using the explicit expression (10.23) for $\psi(\mathbf{x})$ on the surface S, and we shall also add a term consisting of $\phi(\mathbf{x}')$, which is identically zero on the surface S because of the Dirichlet boundary conditions (10.13), multiplied by an arbitrary - appropriately chosen - function:

$$-\int_{S_1} f_\infty(\hat{\mathbf{x}},\mathbf{y},k)g(\hat{\mathbf{x}})d\hat{\mathbf{x}} = \int_S \left[\frac{e^{ikr}}{4\pi r}\frac{\partial\phi(\mathbf{x}')}{\partial n} - \phi(\mathbf{x}')\frac{\partial}{\partial n}\left(\frac{e^{ikr}}{4\pi r}\right)\right] d\mathbf{x}' \ , \quad (10.27)$$

where $r = |\mathbf{x}'|$, so we can identify the term $\frac{e^{ikr}}{4\pi r}$ with the free space Green's function $G(0,\mathbf{x}')$, satisfying $(\nabla^2 + k^2)G(0,\mathbf{x}') = \delta(\mathbf{x}')$.
We shall now see that the integral (10.27) has value unity. This can be shown as follows. First, write the total field in the integral explicitly as the sum of incident and scattered field, $\phi = \phi_i + \phi_s$:

$$\begin{aligned}
&- \int_{S_1} f_\infty(\hat{\mathbf{x}},\mathbf{y},k)g(\hat{\mathbf{x}})d\hat{\mathbf{x}} \\
&= \int_S \left[\frac{e^{ikr}}{4\pi r}\frac{\partial\phi_i(\mathbf{x}')}{\partial n} - \phi_i(\mathbf{x}')\frac{\partial}{\partial n}\left(\frac{e^{ikr}}{4\pi r}\right)\right] d\mathbf{x}' + \int_S \left[\frac{e^{ikr}}{4\pi r}\frac{\partial\phi_s(\mathbf{x}')}{\partial n} - \phi_s(\mathbf{x}')\frac{\partial}{\partial n}\left(\frac{e^{ikr}}{4\pi r}\right)\right] d\mathbf{x}' \\
&= I_i + I_s \qquad (10.28)
\end{aligned}$$

Dealing with the second integral in (10.28), which involves the scattered field ϕ_s, we note that the integral

$$I = \int_{S'} \left[G(0,\mathbf{x}')\frac{\partial\phi_s(\mathbf{x}')}{\partial n} - \phi_s(\mathbf{x}')\frac{\partial G(0,\mathbf{x}')}{\partial n}\right] d\mathbf{x}' \ , \quad (10.29)$$

is invariant with respect to any surface S' on or outside S, by virtue of Green's formula applied to ϕ_s and G in the volume V' bounded by S and S'. Taking S' to be a sphere of large radius R_0, one finds, by using the radiation condition satisfied by ϕ_s, that $I \to 0$ as $R_0 \to \infty$, hence $I_s \equiv 0$.
Dealing with the first integral in (10.28), which involves the incident field ϕ_i, we shall use Green's formula applied to ϕ_i and G, and also use the equation satisfied by G. Thus

$$\begin{aligned}
&\int_S \left[G(\mathbf{x},\mathbf{x}')\frac{\partial\phi_i(\mathbf{x}')}{\partial n} - \phi_i(\mathbf{x}')\frac{\partial}{\partial n}G(\mathbf{x},\mathbf{x}')\right] d\mathbf{x}' \\
&= \int_V \left[G(\mathbf{x},\mathbf{x}')\nabla^2\phi_i(\mathbf{x}') - \phi_i(\mathbf{x}')\nabla^2 G(\mathbf{x},\mathbf{x}')\right] d\mathbf{x}' = \int_V G\nabla^2\phi_i - \phi_i(\delta(\mathbf{x}) - k^2 G) \\
&= -\phi_i(0) = -1 \ . \qquad (10.30)
\end{aligned}$$

It follows

$$\int_{S_1} f_\infty(\hat{\mathbf{x}},\mathbf{y},k)g(\hat{\mathbf{x}})d\hat{\mathbf{x}} = 1 \quad (10.31)$$

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for all directions of incidence $\mathbf{y}$.

The problem is now specified by the two identities (10.31) and (10.23), which can be used to determine first $g(\hat{\mathbf{x}})$, then r_s.
Colton & Monk (1987) accordingly formulate the optimization problem to minimise

$$\sum_{n=1}^{N} \left| \int_{S_1} f_\infty(\mathbf{y}_n, \hat{\mathbf{x}}, k) g(\hat{\mathbf{x}}) ds(\hat{\mathbf{x}}) - 1 \right|^2 \tag{10.32}$$

with respect to $g(\hat{\mathbf{x}})$ from a suitable function class, for a finite number N of incident directions. Given g, hence ψ from equation (10.26), the second identity (10.23) leads to the optimization problem of minimizing

$$\int_{S_1} \left| \psi\left(r_s(\hat{\mathbf{x}})\right) + \frac{e^{ikr_s}}{4\pi r_s} \right|^2 ds(\hat{\mathbf{x}}) \tag{10.33}$$

with respect to $r_s(\hat{\mathbf{x}})$ from a suitable function class.
The estimate for r_s gives an approximation to the surface S.

Colton & Monk (1987) give results for several axially symmetric problems, using trial functions in the form of Fourier series in the azimuthal angle. Their results give excellent reconstructions for a variety of shapes, such as the oblate spheroid, the "peanut" shape, and the "acorn" shape.

The linear sampling method

This method, first proposed by Colton and Kirsch (1996), is also based on the properties of the far field pattern used in the previous section. The basic idea behind it, as its name suggests, is to choose points z in a range known to include the scatterer and provide a scheme for deciding whether the point is on the scatterer. It allows scatterers which are not simply connected, either penetrable or impenetrable, and - when impenetrable - does not require knowledge of the boundary conditions. It was originally started by numerical observations about the behaviour of the kernel of the integral equation for the far field of a point source, and has been successfully applied in a variety of cases. Its mathematical basis hasn't yet been rigourously proven, but we can see how it can be heuristically justified with relatively simple observations.

The linear sampling method therefore also makes use of the Far Field Operator defined earlier:

$$(Fg)(\hat{\mathbf{x}}) = \int_{S_1} f_\infty(\hat{\mathbf{x}}, \mathbf{y}, k) g(\mathbf{y}) ds(\mathbf{y}) \ , \tag{10.34}$$

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where $f(\hat{\mathbf{x}}, \mathbf{y}, k)$ is the measured (known) far field, $g(\mathbf{y})$ is a suitably well-behaved function on S_1, but S_1 is now the unit sphere centred on an *arbitrary* point $\mathbf{z}$.
It is then concerned with solving the integral equation

$$(Fg)(\hat{\mathbf{x}}) = G_\infty(\hat{\mathbf{x}}, \mathbf{z}) \ . \tag{10.35}$$

where F is the Far Field Operator, and $G_\infty(\hat{\mathbf{x}}, \mathbf{z})$ is the far field pattern

$$G_\infty(\hat{\mathbf{x}}, \mathbf{z}) = \frac{1}{4\pi} e^{ik\hat{\mathbf{x}}\cdot\mathbf{z}} \tag{10.36}$$

of a point source centred at $\mathbf{z}$:

$$G(\mathbf{x}, \mathbf{z}) = \frac{1}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{z}|}}{|\ \mathbf{x} - \mathbf{z}\ |} \ , \ \mathbf{x} \neq \mathbf{z} \ , \tag{10.37}$$

and we assume $\mathbf{z}$ is on the scatterer S.

Under the assumption that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in the domain D enclosed by the surface of the scatterer $\partial D = S$ (i.e. not an eigenvalue of the internal Dirichlet problem, as before), it can be shown that
Theorem: *if* $\mathbf{z} \in D$, $\forall \epsilon > 0 \ \exists$ *a solution* $g_\epsilon(\mathbf{z})$ *of the inequality*

$$\|Fg - G_\infty\| \leq \epsilon \tag{10.38}$$

such that

$$\|g_\epsilon(\mathbf{z})\| \to \infty \ \text{and} \ \|v_g(\epsilon, \mathbf{z})\| \to \infty \ \text{as} \ \mathbf{z} \to \mathbf{z}^* \in \partial D \ , \tag{10.39}$$

where $v_g(\epsilon, \mathbf{z})$ *is the Herglotz function with kernel* $g_\epsilon(\mathbf{z})$.

This theorem effectively states that, whatever the incident field on a scatterer D, the far field is arbitrarily close to the far field of a point source centred in D, thus providing a way of calculating the Herglotz kernel g that relates the far field to the incident field. It further provides a way of identifying the surface of the scatterer D as the region where $\|g_\epsilon(\mathbf{z})\|$ becomes unbounded.

The linear sampling method consists in the following steps:

First, find a g that minimises

$$\|Fg - G_\infty\|$$

This problem is ill-posed, and a solution therefore is found by regularising and minimising

$$||Fg - G_\infty||^2 + \alpha||g||^2 \tag{10.40}$$

The solution is sought for each $\mathbf{z}$ on a grid covering some area where we search for the unknown scatterer, hence the name 'sampling method'.

Secondly, the unknown boundary of the scatterer ∂D is then found by looking for the points $\mathbf{z}$ where $\|g_\epsilon(\mathbf{z})\|$ begins to increase sharply.

There are mathematical difficulties with this scheme, because in fact a solution to the minimisation problem is not guaranteed to exist if $\mathbf{z}$ is not on ∂D. To avoid these problems, a modified scheme has been proposed, which factorises the operator F using self-adjoint operators defined on ∂D (*'factorisation method'*, Kirsch 1998). Other modified schemes are also in use.

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10.2 Recovering the refractive index

When the scattering is sufficiently weak, the inverse scattering problem can be linearised and solved using the first Born (or Rytov) approximation (see Chapter 4, were these approximations are introduced for the direct scattering problem). In this case, the (known) scattered field is written as the first Born (or Rytov) solution of the direct scattering problem, then the Fourier transform of the scattered field is related to the Fourier transform of the 'scattering potential' of the object, or medium, thus formally solving the inverse problem.

We shall consider first the Born approximation. We recall (see Ch. 4) that, given some inhomogeneity with refractive index $n(\mathbf{r})$, and a non-scattering background with refractive index 1, the the total field satisfies

$$\nabla^2\psi + k^2(\mathbf{r})\psi = 0 \ , \tag{10.41}$$

where

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0(1 + n_\delta(\mathbf{r})) \ , \tag{10.42}$$

We shall assume $n_\delta(\mathbf{r}) \ll 1$. Substituting $k_0 n(\mathbf{r})$ into (10.41) we get:

$$\nabla^2\psi + k_0^2(\mathbf{r})\psi = -k_0^2(n^2(\mathbf{r}) - 1)\psi \equiv -V(\mathbf{r})\psi \ , \tag{10.43}$$

and the scattered field is then given by

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ . \tag{10.44}$$

The total field is then given by

$$\psi = \psi_i(\mathbf{r}) + \int G(\mathbf{r},\mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ , \tag{10.45}$$

and the scattered field can be approximated first Born approximation by

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')[V(\mathbf{r}')\psi_i(\mathbf{r}')]d\mathbf{r}' \ , \tag{10.46}$$

Here $G(\mathbf{r},\mathbf{r}')$ is the free space Green's function in 3 dimension, i.e.

$$G(\mathbf{r},\mathbf{r}') = \frac{e^{ik_0 r}}{r} \ , \tag{10.47}$$

where $r = |\mathbf{r} - \mathbf{r}'|$. Following Wolf (1969), we shall now use a variant of the Weyl representation of a spherical wave as a superposition of plane waves:

$$G(\mathbf{r},\mathbf{r}') = \frac{ik_0}{2\pi}\int\int_{-\infty}^{\infty}\frac{1}{m}e^{ik_0[p(x-x')+q(y-y')+m(z-z')]}dpdq \ , \tag{10.48}$$

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where:

$$m = (1 - p^2 - q^2)^{1/2} \text{ when } (p^2 + q^2) < 1 \quad (10.49)$$
$$m = i(p^2 + q^2 - 1)^{1/2} \text{ when } (p^2 + q^2) > 1 \quad (10.50)$$

If we now substitute this expression for the Green's function into equation (10.46), we obtain:

$$\psi_s(\mathbf{r}) = \int\int_{-\infty}^{\infty} A^{(\pm)}(p, q; p_0, q_0) e^{ik_0(px+qy\pm mz)} dpdq \ , \quad (10.51)$$

where

$$A^{(\pm)}(p, q; p_0, q_0) = -\frac{ik_0}{8\pi^2 m} \int V\mathbf{r}' e^{ik_0[(p-p_0)x'+(q-q_0)\pm(m-m_0)z']} d\mathbf{r}' \ . \quad (10.52)$$

Here, the upper sign(+) applies in the region $\mathcal{R}^+$ where $z - z' > 0$, and the lower one (-) in the region $\mathcal{R}^-$ where $z - z' < 0$. Equation (10.51) represents the scattered field as an angular spectrum of plane waves, and the spectral amplitude function $A^{(\pm)}(p, q; p_0, q_0)$ is expressed in term of the scattering potential by (10.52). For homogeneous waves, i.e. when m is real, we obtain the relation:

$$A^{(\pm)}(p, q; p_0, q_0) = -\frac{ik_0}{8\pi^2 m} \hat{V}[k_0[(p - p_0), k_0(q - q_0), k_0(\pm m - m_0)] \ , \quad (10.53)$$

where $\hat{V}$ is the Fourier inverse of V:

$$\hat{V}(u, v, w) = \frac{1}{(2\pi)^3} \int V(x, y, z,) e^{ik_0(ux+vy+wz)} dxdydz \ . \quad (10.54)$$

Consider now the scattered field ψ_s in two fixed planes $z = z^+$ and $z = z^-$, situated respectively in $\mathcal{R}^+$ and $\mathcal{R}^-$.

Now, by taking the inverse Fourier transform of (10.51), with z at the fixed values z^+ and z^-, we obtain

$$A^{(\pm)}(p, q; p_0, q_0) = k_0^2 e^{\mp ik_0 m z^\pm} \hat{\psi}_s(k_0 p, k_0 q, z^\pm) \ , \quad (10.55)$$

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where

$$\hat{\psi}_s(u,v,z^{\pm}) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty} e^{-i(ux+vy)}dxdy \qquad (10.56)$$

is the inverse Fourier transform of ψ_s with respect to the variables x and y. Now, comparing (10.55) and (10.53), and using $m = (1-p^2-q^2)^{1/2}$, we obtain

$$\hat{V}(u',v',w'^{\pm}) = \frac{iw}{\pi}e^{\mp iwz^{\pm}}\hat{\psi}_s(u,v,w^{\pm}) \ , \qquad (10.57)$$

where

$$\begin{aligned} u' &= u - k_0p_0 \\ v' &= v - k_0q_0 \\ w' &= \pm w - k_0m_0 \end{aligned} \qquad (10.58)$$

and

$$w = (k_0^2 - u^2 - v^2)^{1/2} \ . \qquad (10.59)$$

Equation (10.57) shows that some of the three-dimensional Fourier components of the scattering potential v, and therefore the unknown refractive index, can be immediately determined by the two-dimensional components of the scattered field in the two planes $z = z^+$ and $z = z^-$.

Note that (10.57) is valid only for those two-dimensional Fourier components of $\hat{\psi}_s$ and ψ_s about which the information is carried by homogeneous waves, i.e. those for which $u^2 + v^2 \leq k_0^2$. In general, it is impossible to reconstruct inverse data associated with the high spectral components for which the information is carried by evanescent waves, because these waves decay very rapidly from the scatterer and do not contribute to the far field. This limitation arises because the problem is ill-posed.
We saw earlier that one way to obviate the limitations caused by ill-posedness is to use the Tikhonov (or other) regularization. In this case then, if we represent by $\mathbf{A}$ the integral operator in (10.46):

$$\mathbf{A}V(\mathbf{r}) = \int G(\mathbf{r},\mathbf{r}')V(\mathbf{r}')\psi_i(\mathbf{r}')d\mathbf{r}' \ , \qquad (10.60)$$

then the problem we need to solve is

$$\mathbf{d} = \mathbf{A}V(\mathbf{r}) \qquad (10.61)$$

where $\mathbf{d}$ is the vector of the scattered field measurements. This can be regularised by minimising the Tikhonov functional

$$J_\alpha = \| \mathbf{A}V(\mathbf{r}) - \mathbf{d} \|^2 + \alpha \| V(\mathbf{r}) \|^2 \qquad (10.62)$$

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with the penalty parameter α usually chosen based on knowledge of the noise level.

Using the first Born approximation for the inverse scattering problem reduces the non-linear inverse problem to a completely linear one.
We can retain some non-linearity either by adding higher order terms in the Born approximation, or by using the ***distorted-wave Born approximation*** (DWBA). In the DWBA, instead of approximating the 'zero order' solution with the incident field as in the first Born illustrated above, we start with a perturbed field, in other words, instead of writing the refraction index as

$$n(\mathbf{r}) = 1 + n_\delta \ , \tag{10.63}$$

we write

$$n^2(\mathbf{r}) = n_0^2(\mathbf{r}) + \epsilon n_1 + \epsilon^2 n_2 + \dots \tag{10.64}$$

The DWBA then is obtained by seeking a solution of the Helmholtz equation (10.41) in the form:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \epsilon\psi_1(\mathbf{r}) + \dots \tag{10.65}$$

The solution terms in this series can be computed by solving:

$$\begin{aligned}
(\nabla^2 + k_0^2 n_0^2 \mathbf{r})\psi_0 &= 0 \\
(\nabla^2 + k_0^2 n_0^2 \mathbf{r})\psi_1 &= -k_0^2 n_1 \psi_0 \\
(\nabla^2 + k_0^2 n_0^2 \mathbf{r})\psi_2 &= -k_0^2 n_2 \psi_0 - k_0^2 n_1 \psi_1 \\
&\dots
\end{aligned} \tag{10.66}$$

So the integral equation corresponding to (10.46) is now:

$$\psi_s(\mathbf{r}) = \int G^{(k)}(\mathbf{r}, \mathbf{r}')[V(\mathbf{r}')\psi_i(\mathbf{r}')]d\mathbf{r}' \ , \tag{10.67}$$

and $G^{(k)}(\mathbf{r}, \mathbf{r}')$ is not the free space Green's function any more. If $n_0^2(\mathbf{r}) = 1$, then the DWBA coincides with the Born approximation. In the DWBA it is also possible of course to go to higher terms and include more iterations.
It should be noted, though, that in general, if the measured data is contaminated with noise, so that the actual total field ψ^a is:

$$\psi^a(\mathbf{r}) = \psi(\mathbf{r}) + \Delta(\mathbf{r}) \ , \tag{10.68}$$

where $\Delta(\mathbf{r})$ is the noise, then

$$\psi_s(\mathbf{r}) = \psi(\mathbf{r}) - \psi_i(\mathbf{r}) + \Delta(\mathbf{r}) \ . \tag{10.69}$$

Hence, as successive iterations improve on $\psi_i(\mathbf{r})$ so that it is closer to $\psi(\mathbf{r})$, $\psi_s(\mathbf{r})$ is swamped by noise. Other variants of the Born iterative method are more robust and also less time-consuming, especially in higher dimensions.

10.3 Iterative methods

Iterative methods have been used in inverse scattering problems both for shape reconstruction and for recovering the refractive index of a medium. They are particularly useful for non-linear inverse problems, where minimisation of the Tikhonov functional J_α can be problematic, as J_α is not strictly convex in these cases.

Many iterative methods are very computationally expensive, requiring a large number of iterations and often the calculation of derivatives at each step; some also have large memory requirements. Nevertheless, several refinements of computationally expensive methods have been proposed, and some iterative schemes which are efficient do exist, making them a very useful tool for achieving accurate solutions in many cases. In recent years several rigorous results about convergence have also been achieved. (Refs ...)

If we, as usual, formally state the inverse problem as, given an operator A and known data y, find x s.t. $Ax = y$, then the idea of an iterative method is to start with an initial guess x_0 and define an operator F and iterative strategy $F(x_n, y) = x_{n+1}$ which, in the limit $n \to \infty$, tends to the fixed point equation $F(x, y) = x$.

Typically rather than solving $Ax = y$ we shall need to find the generalised solution $x^\dagger$ which the unique solution of

$$A^*Ax = A^*y \; . \tag{10.70}$$

There are of course several possible choices of fixed point operator F. A natural choice, for example, is to re-write (10.70) as $A^*y - A^*Ax - x = x$, leading to the iteration

$$x_{n+1} = x_n - A^*(Ax_n - y) \; . \tag{10.71}$$

This is **Landweber iteration**. The quantity in brackets on the right measures the error in the approximate solution x_n. It is called the ***residual***, and is usually denoted by r_n. With this notation, Landweber iteration (10.71) becomes $x_{n+1} = x_n - A^*r_n$.

All iterative schemes can be written conveniently as:

$$x_{n+1} = x_n + \beta_n d_n \; , \tag{10.72}$$

where d_n is a vector giving the direction at the n-th iteration, and is some function of the residual, and β_n is a scalar giving the step-size along d_n.

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The Landweber iteration scheme derived above then corresponds to (10.72) with:

- $\beta_n = 1$, $d_n = -r_n$

Some other common iterative schemes are obtained as follows:

- $\beta_n = \frac{(r_n,r_n)}{(r_n,Ar_n)}$, $d_n = -r_n$ (steepest descent)
- $\beta_n = -\frac{(d_n,r_n)}{(d_n,Ad_n)}$, with $d_0 = -r_0$ and $d_{n+1} = r_{n+1} + \frac{(r_{n+1},r_{n+1})}{(r_n,r_n)}d_n$ (conjugate gradient)

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11 References and further reading

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Some background, and reference for applications

The following four are all very good and comprehensive standard graduate textbooks on waves, both acoustic and electromagnetic, and include propagation, scattering, noise generation and mathematical methods :

M. Born and E. Wolf *Principles of Optics*, Pergamon 1970

D.S. Jones *Acoustic and Electromagnetic Waves* Clarendon Press 1986

P.M. Morse and K. U. Ingard *Theoretical Acoustics*, McGraw-Hill 1968

A.D. Pierce *Acoustics* ASA, 1989.

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A good book for complex methods, and specifically also topics of relevance to wave scattering, such as asymptotic approximations and the Wiener-Hopf method, is:
M.J. Ablowitz and A.S. Fokas *Complex variables*, CUP, 1997

More on asymptotic Methods: Hinch *Perturbation Methods*, Cambridge: CUP 1991

The following book gives a very useful compendium of the main known solutions to 'canonical' scattering problems (including the asymptotic solutions) for both acoustic and electromagnetic waves:
J.J. Bowman, T.B.A. Senior and P.L.E Uslenghi *Electromagnetic and acoustic scattering by simple shapes*, North-Holland, 1969

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