Wave Propagation and Scattering
Mathematical Tripos Part III

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Examples Sheet 1

The sign convention $\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ *is used here, with inverse Fourier transforms having a factor of* $1/2\pi$*. Three dimensions should be assumed unless specified otherwise. Please send any comments and corrections to* `ejb48@cam.ac.uk`.

1. A plane wave of amplitude A travelling in a direction $\boldsymbol{d}$ is given by $\rho' = A\cos\big(\omega(t - \boldsymbol{x}\cdot\boldsymbol{d}/c_0)\big)$, where $|\boldsymbol{d}| = 1$. Find the energy density E and the energy flux $\boldsymbol{I}$. If $\langle\phi\rangle$ denotes the time average of ϕ, show that

$$\langle \boldsymbol{I}\rangle = \langle E\rangle c_0 \boldsymbol{d};$$

i.e. on average, energy flows in the direction $\boldsymbol{d}$ at speed c_0.

2. Solve the wave equation outside a bubble of radius $a + \varepsilon\cos(\omega t)$, where ε is small. Calculate the average energy density $\langle E\rangle$ and the average energy flux $\langle \boldsymbol{I}\rangle$. What does the quantity $|\langle \boldsymbol{I}\rangle|/(c_0\langle E\rangle)$ tell you about the efficiency of an oscillating bubble as a sound producer?

3. *The 1D Green's function.*

(a) Show that

$$f(x) = \begin{cases} 1 & |x| < a \\ 0 & \text{otherwise} \end{cases} \qquad \Rightarrow \qquad \tilde{f}(k) = \frac{2}{k}\sin(ak),$$

where $\tilde{f}(k)$ is the Fourier transform of $f(x)$. Hence, or otherwise, show that

$$\int_0^\infty \frac{1}{x}\sin(mx)\,\mathrm{d}x = \frac{\pi}{2}\operatorname{sgn} m.$$

(b) Show that the 1D Green's function for the wave equation is

$$G(x,t;\,y,\tau) = \frac{1}{2c_0}H\big(c_0(t-\tau) - |x-y|\big),$$

where H is the Heaviside step function.

4. *The 2D Green's function.* Use the 3D wave equation Green's function to calculate the wave field generated by an instantaneous line source in the z direction,

$$\frac{\partial^2 G}{\partial t^2} - c_0^2\nabla^2 G = \delta(x-x_0)\delta(y-y_0)\delta(t-t_0).$$

Identify this solution as the 2D wave equation Green's function.

5. *Helmholtz equation Green's functions.*

(a) Derive the Green's function for the 1D Helmholtz equation in free space directly, and verify it agrees with the Fourier transform of the 1D wave equation (time-dependent) Green's function.

(b) Derive the Green's function for the 3D Helmholtz equation in free space directly, and verify it agrees with the Fourier transform of the 3D wave equation (time-dependent) Green's function.

Hint: You may like to assume the 3D Green's function is spherically symmetric about the point source, then integrate the differential equation over a ball of radius ε *and consider* $\varepsilon \to 0$.

It is interesting to note that the 2D Green's function for the Helmholtz equation is given by $\frac{\mathrm{i}}{4}H_0^{(2)}(k_0|\boldsymbol{x}-\boldsymbol{y}|)$, where $H_0^{(2)}$ is the 2nd Hankel function, related to Bessel functions by $H_0^{(2)}(z) = J_0(z) - \mathrm{i}Y_0(z)$. If you're feeling keen you might like to try proving this (for Bessel function identities see Abramowitz & Stegun, chapter 9, available online).

6. A lecture theatre has a good design so that the lecturer's voice is well projected to the audience and the sound of the audience is damped by overhead absorbing panels. Assuming all speak equally loudly, compare the volumes of: (i) the lecturer heard by a student at the back; (ii) a student at the back heard by a student at the front; (iii) a student at the back heard by the lecturer.

7. *Quadrupoles.* By considering two dipoles with source strengths $\boldsymbol{d}(t)$ and $-\boldsymbol{d}(t)$ placed at $-\boldsymbol{\ell}/2$ and $\boldsymbol{\ell}/2$ respectively, show that the resulting wave field is given by

$$\rho'(\boldsymbol{x},t) \sim \frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{t_{ij}(t-|\boldsymbol{x}|/c_0)}{4\pi c_0^2|\boldsymbol{x}|}\right), \qquad \text{where} \quad t_{ij}(t) = \ell_j d_i(t).$$

Sketch the location and orientation of the two dipoles, and the locations of the four underlying monopoles, in the following two cases. In which directions do they radiate most strongly?

(a) $t_{ij} = \begin{pmatrix} e^{i\omega t} & 0 \\ 0 & 0 \end{pmatrix}$ a "longitudinal" quadrupole

(b) $t_{ij} = \begin{pmatrix} 0 & e^{i\omega t} \\ 0 & 0 \end{pmatrix}$ a "lateral" quadrupole

(c) $t_{ij} = \begin{pmatrix} 0 & 0 \\ e^{i\omega t} & 0 \end{pmatrix}$ a second "lateral" quadrupole

8. A point mass source of strength $q(t)$ is in uniform motion with velocity $c_0 M$ in the 1 direction in 3D in a fluid otherwise at rest. An observer stands at a fixed location in the 1, 2-plane, giving a distance $r(t)$ and angle $\theta(t)$ from source to observer, with $\theta = 0$ being directly ahead of the source.

(a) Show that the sound heard by the observer is given by

$$\rho' = \frac{\partial}{\partial t}\left(\frac{q(\tau)}{4\pi c_0^2 r(\tau)|1-M_r(\tau)|}\right),$$

where $M_r(\tau) = M\cos\big(\theta(\tau)\big)$ is the Mach number of the point source in the direction of the observer, and τ satisfies $r(\tau) = c_0(t-\tau)$.

(b) Show that

$$\frac{\partial \tau}{\partial t} = \frac{1}{1-M_r(\tau)},$$

thereby identifying $\big(1-M_r(\tau)\big)^{-1}$ as the Doppler shift factor. Solve for $\tau(t)$ in the case $|M|<1$, and hence show that

$$\frac{\partial \tau}{\partial t} = \frac{1}{1-M^2}\left(1+\frac{M\cos\big(\theta(t)\big)}{\sqrt{1-M^2\sin^2\big(\theta(t)\big)}}\right).$$

How would the sound heard by the observer be different if $|M|>1$?

9. **(a)** For any Green's function $\tilde{G}(\boldsymbol{x};\boldsymbol{y})$ of the Helmholtz equation satisfying $\nabla^2\tilde{G} + k_0^2\tilde{G} = -\delta(\boldsymbol{x}-\boldsymbol{y})/c_0^2$ within a fixed (i.e. stationary) volume V with outward normal $\boldsymbol{n}$, show that

$$\tilde{\rho'}(\boldsymbol{x})\mathbf{1}_{\boldsymbol{x}\in V} = \int_V \tilde{T}_{ij}(\boldsymbol{y})\frac{\partial^2\tilde{G}(\boldsymbol{y};\boldsymbol{x})}{\partial y_i \partial y_j}\,\mathrm{d}\boldsymbol{y} - \int_{\partial V}\{\tilde{p}n_i - \tilde{\sigma}_{ij}n_j\}(\boldsymbol{y})\frac{\partial \tilde{G}(\boldsymbol{y};\boldsymbol{x})}{\partial y_i}\,\mathrm{d}S(\boldsymbol{y}),$$

assuming only that $u_j n_j = 0$ on ∂V and that $k_0 \neq 0$, where $\mathbf{1}_P$ is 1 if P is true and 0 otherwise and $\tilde{}$ denotes the Fourier transform in time. *Hint: consider* $\boldsymbol{\nabla}\cdot(\tilde{G}\boldsymbol{\nabla}\tilde{\rho}' - \tilde{\rho}'\boldsymbol{\nabla}\tilde{G})$ *and integrate over* V.

(b) Now let V be the fluid outside a rigid fixed object, and neglect viscosity. Let $\tilde{G}_0$ be the free field Green's function as derived in question 5(b), and let $\tilde{G}_1$ be the Green's function satisfying $\boldsymbol{n}\cdot\boldsymbol{\nabla}\tilde{G}_1 = 0$ on ∂V. Writing $\tilde{G}_1 = \tilde{G}_0 + \tilde{G}_s$, so that $\tilde{G}_s$ represents the scattered acoustic field from the object, show that

$$\int_V \tilde{T}_{ij}(\boldsymbol{y})\frac{\partial^2\tilde{G}_s(\boldsymbol{y};\boldsymbol{x})}{\partial y_i \partial y_j}\,\mathrm{d}\boldsymbol{y} = -\int_{\partial V}\tilde{p}(\boldsymbol{y})n_i\frac{\partial\tilde{G}_0(\boldsymbol{y};\boldsymbol{x})}{\partial y_i}\,\mathrm{d}S(\boldsymbol{y}).$$

Interpret this as saying that the surface integral on the right hand side represents the sound field from sources within the fluid scattering from the object, rather than a separate source of sound on the object's surface. Are these terms dipoles or quadrupoles?

(c) Use scaling arguments to calculate the magnitude of sound radiated to the far field from turbulent flow around a stationary object.