Wave Propagation and Scattering Dr E.J. Brambley
Mathematical Tripos Part III Lent Term 2012

Examples Sheet 2

The sign convention $\exp\{\mathrm{i}\omega t - \mathrm{i}kx\}$ *is used here, with inverse Fourier transforms having a factor of* $1/2\pi$*. Please send any comments and corrections to* `ejb48@cam.ac.uk`*.*

1. Taking a source location $\boldsymbol{y} = \boldsymbol{\eta} + \boldsymbol{X}(\boldsymbol{\eta}, \tau)$, so that $\boldsymbol{\eta}$ is a label for where the source was at time $\tau = 0$, integrate the Ffowcs-Williams–Hawkings equations to show that

$$\begin{aligned} \rho - \rho_0 = {} & \frac{\partial^2}{\partial x_i \partial x_j} \int_{F(\boldsymbol{\eta})>0} \left[\frac{T_{ij}(\boldsymbol{\eta} + \boldsymbol{X})}{4\pi c_0^2 r |1 - M_r|} J_3(\boldsymbol{\eta}) \right]_{\tau=\tau^*} \mathrm{d}V(\boldsymbol{\eta}) \\ & - \frac{\partial}{\partial x_i} \int_{F(\boldsymbol{\eta})=0} \left[\frac{F_i(\boldsymbol{\eta} + \boldsymbol{X})}{4\pi c_0^2 r |1 - M_r|} J_2(\boldsymbol{\eta}) \right]_{\tau=\tau^*} \mathrm{d}S(\boldsymbol{\eta}) \\ & + \frac{\partial}{\partial t} \int_{F(\boldsymbol{\eta})=0} \left[\frac{Q(\boldsymbol{\eta} + \boldsymbol{X})}{4\pi c_0^2 r |1 - M_r|} J_2(\boldsymbol{\eta}) \right]_{\tau=\tau^*} \mathrm{d}S(\boldsymbol{\eta}), \end{aligned} \qquad (*)$$

where $F(\boldsymbol{y}) = f(\boldsymbol{y}, \tau)$ at time $\tau = 0$, $r(\tau) = |\boldsymbol{x} - \boldsymbol{\eta} - \boldsymbol{X}(\boldsymbol{\eta}, \tau)|$, τ^* satisfies $\tau^* = t - r(\tau^*)/c_0$, J_3 is the Jacobian for the transformation from $\boldsymbol{y}$ to $\boldsymbol{\eta}$ space, $J_2 = J_3|\boldsymbol{\nabla}_{\boldsymbol{y}} f|/|\boldsymbol{\nabla}_{\boldsymbol{\eta}} F|$ is effectively the surface area Jacobian for the same transformation, and

$$M_r = \frac{x_i - \eta_i - X_i}{c_0|\boldsymbol{x} - \boldsymbol{\eta} - \boldsymbol{X}|} \frac{\partial X_i}{\partial \tau}$$

is the Mach number of the source in the observer's direction. Why does the quadrupole term in $(*)$ have an extra factor of $|1 - M_r|^{-1}$ when compared with the expression given in lectures?

2. A compact rigid sphere of radius a undergoes small oscillations in the x-direction with velocity $U(t)$ with $|U(t)| \ll c_0$. What does the compact low-Mach-number result from lectures suggest about the monopole radiation in the far field? By assuming the motion locally about the sphere to be incompressible, show that the dipole radiation in the far field is given to leading order by

$$\rho - \rho_0 \sim \frac{\rho_0 a^3}{6c_0^3 r} \ddot{U}(t - r/c_0) \cos\Theta,$$

where the observer is at a distance r from the centre of the sphere at an angle Θ to the x-axis.

3. Given $F(k)$ analytic and nonzero in a strip $-\sigma^+ < \mathrm{Im}(k) < \sigma^-$, consider a multiplicative factorization $F(k) = F^+(k)F^-(k)$ with $F^+(k)$ analytic and nonzero for $\mathrm{Im}(k) > -\sigma^+$ and $F^-(k)$ analytic and nonzero for $\mathrm{Im}(k) < \sigma^-$. Show that there is a unique factorization (up to multiplication by a constant) such that $F^\pm$ have at most algebraic growth or decay as $|k| \to \infty$ in their respective regions, meaning $|k|^{-n} < |F^\pm(k)| < |k|^n$ for all $|k| > M$ for some M and n).

4. (**a**) Give an additive decomposition of both $\cos(z)$ and $\sin(z)/z$.

   (**b**) Give a multiplicative decomposition of $k^4 - k_0^4$ with $-\pi/2 < \arg(k_0) < 0$.

   (**c**) Use the integral formula to find an additive factorization of

$$F(k) = \frac{1}{(k + k_0\cos\theta_0)\sqrt{k - k_0}},$$

   where $\mathrm{Im}(k_0) < 0$ and the branch cut is chosen such that $\sqrt{k - k_0}$ is analytic in the upper half plane. *Hint: Close the contours in the upper half plane for both* F^+ *and* F^-*.*

5. (**a**) Write down the exact solution (in integral form) for the 2D scattering of a plane wave incident at an angle θ_0 to a rigid semi-infinite plate, observed at a distance r and angle θ from the end of the plate. Hence show that the scattered field can be expressed in the far field to leading order as

$$\phi \sim Ae^{-\mathrm{i}k_0 r}\int_{-\infty}^{\infty} \frac{e^{-u^2}\,\mathrm{d}u}{u - e^{3\pi\mathrm{i}/4}\beta} + 2\pi\mathrm{i}H(-\beta)\frac{A\gamma^-(k_0\cos\theta)}{\gamma^-(-k_0\cos\theta_0)} \exp\left\{\mathrm{i}k_0 r(\cos\theta\cos\theta_0 - |\sin\theta\sin\theta_0|)\right\},$$

where

$$A = \frac{\mathrm{sgn}(\sin\theta)k_0\sin\theta_0}{2\pi\gamma^+(-k_0\cos\theta_0)\gamma^-(k_0\cos\theta)}, \qquad \text{and} \qquad \beta = \frac{\cos\theta + \cos\theta_0}{|\sin\theta|}\sqrt{\tfrac{1}{2}rk_0}.$$

(**b**) Show that

$$I(\lambda) = \int_{-\infty}^{\infty} \frac{e^{-u^2\lambda}}{u - e^{3\pi\mathrm{i}/4}\beta}\,\mathrm{d}u \qquad \Rightarrow \qquad \frac{\mathrm{d}I}{\mathrm{d}\lambda} - \mathrm{i}\beta^2 I = -e^{3\pi\mathrm{i}/4}\beta\sqrt{\frac{\pi}{\lambda}},$$

and hence that

$$I(1) = \mathrm{i}\pi\mathrm{sgn}(\beta)e^{\mathrm{i}\beta^2}\mathrm{erfc}\big(e^{\mathrm{i}\pi/4}|\beta|\big), \qquad \text{where} \quad \mathrm{erfc}(z) = 1 - \mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,\mathrm{d}t.$$

(**c**) Now let $r \to \infty$ and $\cos\theta \to -\cos\theta_0$ such that β remains fixed. In this limit, show that

$$\mathrm{i}k_0 r(\cos\theta\cos\theta_0 - |\sin\theta\sin\theta_0|) = -\mathrm{i}k_0 r + \mathrm{i}\beta^2 + O\big(r^{-1/2}\big),$$

and hence that

$$\phi \sim 2\pi\mathrm{i}Ae^{-\mathrm{i}k_0 r + \mathrm{i}\beta^2}\Big(1 - \tfrac{1}{2}\,\mathrm{erfc}\big(e^{-3\pi\mathrm{i}/4}\beta\big)\Big).$$

Interpret this as showing that the geometric optic pole "turns on" smoothly over an angle of order $(rk_0)^{-1/2}$; this solution is the Fresnel field mentioned in lectures.

6. Consider a thin plate at $y = 0, x < 0$ undergoing small amplitude oscillations with normal velocity $V_0\exp\{\mathrm{i}\omega t - \mathrm{i}qx\}$. Find an expression for the generated fluid perturbation and calculate the sound radiated to the far field. *Hint: take* $\mathrm{Im}(\omega) < 0$ *and* $\mathrm{Im}(q) > 0$, *apply the Wiener–Hopf technique, then let the imaginary parts tend to zero.*

7. A loudspeaker causes a velocity in the x-direction of $U(t)$ at $x = 0$, with $\varepsilon = \max U/c_0 \ll 1$. By considering the $O(\varepsilon)$ and $O(\varepsilon^2)$ terms in the asymptotic expansion of the x-velocity $u(x,t)$, show that

$$u(x,t) = U(\tau) + x\left[\frac{\gamma+1}{2c_0^2}U(\tau)U'(\tau) + \frac{\frac{4}{3}\mu + \mu_B + \frac{\gamma-1}{c_p}\kappa}{2\rho_0 c_0^3}U''(\tau)\right] + O\big(U^3, \mu U^2, \kappa U^2\big),$$

where $\tau = t - x/c_0$. Hence show that this asymptotic expansion breaks down when x is of the order of λ/ε, where λ is a typical wavelength.

8. Solve the inviscid Burgers' equation with initial condition $f(0,\theta) = (1 - |\theta|)$ for $|\theta| < 1$ and $f(0,\theta) = 0$ otherwise. At what value of Z does a shock form, and what is the solution for larger Z than this?

9. A travelling wave solution to the viscous Burgers' equation is a solution of the form $f(Z,\theta) = q(\theta + VZ)$ with V constant. If $f(0,\theta) \to q_-$ as $\theta \to -\infty$ and $f(0,\theta) \to q_+$ as $\theta \to +\infty$, find all possible travelling wave solutions. What happens as the dissipation $\alpha \to 0$.

10. Let $f(0,\theta) = A\delta(\theta)$. By using the Cole–Hopf transform of the viscous Burgers' equation, find $f(Z,\theta)$, and hence show that, for $2\alpha/A \gg 1$,

$$f(Z,\theta) \sim \frac{A}{\sqrt{4\pi\alpha Z}}e^{-\theta^2/4\alpha Z}.$$

What is the corresponding result for $2\alpha/A \ll 1$?

11. A rectangular waveguide in the x-direction has width W and height H. The two side walls and the bottom wall are hard and do not move, while the top boundary is "soft", meaning it cannot support a pressure different from ambient, and hence $\tilde{p} = 0$ there. There is a subsonic uniform mean flow of velocity U_0 along the waveguide.

(**a**) Solve for the wave modes in the waveguide, and hence show that waves can propagate along the waveguide if and only if

$$\omega > \frac{\pi}{2H}\sqrt{c_0^2 - U_0^2}.$$

(**b**) Now consider a square duct ($H = W$) with no mean flow ($U_0 = 0$). At $x = 0$ a piston covering the whole waveguide cross-section oscillates in the x-direction with small amplitude and a frequency such that the diagonal of the waveguide cross-section is exactly one wavelength (i.e. $2\pi c_0/\omega = H\sqrt{2}$). Sufficiently far from the piston that evanescent waves can be neglected, find the radiated pressure.