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## Examples Sheet 3

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1. Consider a time-harmonic, monochromatic plane wave in 2D, incident upon an infinite, flat, perfectly reflecting surface at $z = 0$. Write the formal solution for the scattered field using the integral form of the Helmholtz equation (Kirchhoff-Helmholtz equation) showing explicit dependance on the reflection coefficient $R$ in the case of:

   a) Dirichlet boundary conditions;

   b) Neumann boundary conditions.

   Comment on the 'locality' of the solution.

2. Consider a time-harmonic, monochromatic plane wave incident at an angle $\theta$ onto an interface which has surface impedance $Z$. Writing the total field above the interface as the sum of an incident and a reflected field, express the reflection coefficient in terms of the impedance.

   (a) Write the mean (time-averaged) energy flux across the surface in terms of the impedance, and find a condition on the real part of the impedance which holds if the surface absorbs energy.

3. Consider a time-harmonic, monochromatic wave propagating in an isotropic and inhomogeneous medium with refractive index $n = n(z)$. Assume that the wavenumber $k(z)$ is given as a function of a known medium profile as

$$k^2(z) = k_0^2 n^2(z) = k_0^2 \left( a^2 - \frac{b^2}{(z - z_0)^2} \right) , \qquad (1)$$

   where $a$, $b$ and $z_0$ are arbitrary constant. If the wave is obliquely incident upon the medium: $\psi(x, z) = u(z) \exp(-ik_0 \sin\theta_0 x)$ Derive the solution $u(z)$.

   *[Use the known result that the differential equation*

$$\frac{d^2W}{dz^2} + \left( \beta^2 - \frac{4\nu^2 - 1}{4z^2} \right) W = 0 \qquad (2)$$

   *has solution*

$$W(z) = \sqrt{z} Z_\nu(\beta z) ,$$

   *where $Z_\nu(\beta z)$ is a Bessel function.]*

4. Consider a time-harmonic point source in a 3D refractive medium with azimuthal symmetry, so that we can use cylindrical coordinates $(r, z, \theta)$, and the total field $\psi$ can be written as

$$\psi(r, z) = \frac{u(r, z)}{\sqrt{r}} , \qquad (3)$$

   Given the reference wave number $k_0 = \omega/c_0$ and the index of refraction of the medium $n(r, z) = c_0/c(r, z)$,

   a) write the Helmholtz equation for the wave $u(r, z)$ in the far field (large $r$);

   b) derive a factorisation of this Helmholtz equation in the case where the refractive index is a function of $z$ only: $n = n(z)$, using the operators

$$A = \frac{\partial}{\partial r} , \quad B = \sqrt{\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2} , \qquad (4)$$

   c) derive a parabolic equation in the paraxial approximation for the reduced wave

$$E = u(r, z) e^{-ikr}$$

   *(select only outgoing waves, and use Taylor expansion)*

5. Consider a time-harmonic plane wave in the plane $x - z$, incident from $z > 0$ upon the interface $z = 0$ between two homogeneous half-spaces, with wave speed $c_0$ in the upper medium ($z > 0$) and $c_1$ in the lower medium ($z < 0$), and let $n = c_0/c_1$ be the index of refraction of the lower medium with respect to the upper medium. The relationship between the cosines of the incident and transmitted angles $\theta_i$ and $\theta_T$ is

$$n \cos\theta_T = \left(1 + \frac{n^2 - 1}{\cos^2\theta_i}\right)^{1/2} \cos\theta_i \tag{5}$$

(a) write the following solutions for the reflected and the transmitted field:

(i) the exact solution

(ii) the first Born approximation

(iii) the first Rytov approximation

(b) Assuming that

$$\alpha \equiv \frac{n^2 - 1}{\cos^2\theta_i} < 1 \quad ,$$

it is possible to expand $n\cos\theta_T$ as given by equation (3) above. Using the parameter $\alpha$, compare the Born and Rytov approximations, and comment on the limit $\alpha \to 0$ .

6. Consider the Exercise on p. 25 of the lecture notes, giving a recipe for constructing a continuous, randomly rough surface $h(x)$. Follow the same procedure, but modify the expression for $h(x)$

$$h(x) = \sqrt{\Delta\nu} \sum_{n=1}^{N} A_n \sin(\nu_n x + \phi_n),$$

in order to construct a surface that is not statistically stationary.

7. Let $f(x)$ be a continuous stochastic process which is statistically stationary and with autocorrelation function

$$\rho(\xi) =< f(x) f(x+\xi) > \ .$$

a) Show that

$$\left\langle f(y)\frac{df(x)}{dx}\right\rangle = \left.\frac{d\rho}{d\xi}\right|_{\xi=y-x} .$$

and similarly find an expression for the autocorrelation of the slope

$$\left\langle \frac{df(x_1)}{dx}\frac{df(x_2)}{dx}\right\rangle$$

For what value of $\xi$ does $\rho$ reach a global maximum? What happens as $\xi \to \infty$?

b) If, further, $f(x)$ has normal distribution, mean zero and variance $< (f(x))^2 >= f_0$, find the mean value $< \exp(if(x)) >$.

8. A time-harmonic plane wave $\psi_i = \exp(ik[x\sin\theta - z\cos\theta])$ is scattered by a random rough surface $h(x)$, producing a scattered field $\psi(x, z)$. $h(x)$ is a member of a statistical ensemble which is stationary with respect to translation in $x$, with rms height $< h^2 >= \sigma^2$, autocorrelation function $\rho(\xi)$. Consider the autocorrelation

$$A(\nu' - \nu) = \left\langle \hat{\psi}(\nu')\hat{\psi}^*(\nu)\right\rangle \ , \tag{6}$$

where $\hat{\psi}(\nu)$ is the Fourier transform of $\psi_s$ *along the horizontal mean plane*, $z = 0$:

$$\hat{\psi}(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_s(x, 0)e^{-i\nu x}\, dx \tag{7}$$

a) Write the scattered field $\psi_z(x, z)$ as a superposition of Fourier components $\hat{\psi}(\nu)$.

b) Derive an expression for $A(\nu' - \nu)$ in terms of $m(\xi) =< \psi_s(x)\psi*_s(y) >$ (*field coherence function*), where $\xi = y - x$. Comment on the mean intensity of the scattered field, $m(0)$.

9. Consider the same plane wave incident field $\psi_i = \exp(ik[x \sin\theta - z\cos\theta])$ as in question 8 above. and denote by $\rho$ the density of the medium above the scattering surface.

a) Write an expression for the average energy flux of the incident field across some horizontal plane, in the direction $\mathbf{n} = -z$.

b) Write an expression for the average energy flux of the scattered field across some horizontal plane, in the direction $\mathbf{n} = z$.

c) From conservation of energy, derive a relation between $\cos\theta$ and the so-called angular spectrum, $\mid \hat{\psi}(\nu) \mid^2$.

10. Consider a field $\phi e^{i\omega t}$ in a weakly scattering extended random medium $(x, z)$, where $x$ is horizontal and $z$ is vertical. Let the refractive index be $n(x, z) = 1 + \mu W(x, z)$ where $W$ is statistically stationary, has mean zero, and variance one, with autocorrelation function

$$\rho(\xi, \eta) =< W(x, z)W(x + \xi, z + \eta) >$$

and scale sizes $H$ and $L$ in the horizontal and vertical direction respectively. Suppose that the slowly varying part of the field $E$ obeys the parabolic equation. Over a small distance $d$ the effect of the medium on the field is a phase change $E(x + d, z) = E(x, z)e^{i\phi}$.

(a) Show that $\phi$ can be written

$$\phi(z) = k_0\mu \int_x^{x+d} W(x', z)dx'$$

and find its variance $< \phi^2 >$.

(b) Describe the effect on the scattering of 'stretching' each of the scale sizes $H$ and $L$.