Wave Propagation and Scattering Dr O. Rath Spivack
Mathematical Tripos Part III Lent Term 2012

Examples Sheet 4

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1. Consider a time-harmonic, monochromatic plane wave in a 2-dimensional space (x, z), incident at an angle θ_1 upon an impedance surface defined by the x-axis: $z = 0$, which divides the space into an upper medium with density ρ_1, wavespeed c_1 and corresponding wavenumber $k_1 = \omega/c_1$, and a lower medium with density ρ_2, wavespeed c_2 and corresponding wavenumber $k_2 = \omega/c_2$. Suppose that a perfectly reflecting surface is placed at $z = -d$, and impose Dirichlet boundary conditions at the surface $z = -d$.

   Derive an expression for the total acoustic field at an arbitrary point $(x, -d < z < 0)$ in medium 2.

2. Consider a 3-dimensional, random medium with refractive index
$$n = 1 + \mu W(x, y, z) \ ,$$
   where μW is a normally distributed random correction to the refractive index, such that $< W > \ = 0$ and $< W^2 > \ = 1$, and $\mu \ll 1$ is a small constant. Consider a time-harmonic field which is assumed to be propagating at a small angle to the horizontal x.

   (a) derive the equation giving the evolution along x of the first moment $< E >$, where E is the reduced wave.

   (b) Assume the medium is statistically stationary in y and z. Find the average field at an arbitrary point (x, y, z) due to a plane wave propagating in the positive x direction and defined by $E(x, y, z) = E_0$ at $x = 0$.

   (c) What happens if the field is a superposition of plane waves, for example a Gaussian beam?

3. Given the Helmholtz equation for a wave $\psi(x, y)$: $(\nabla^2 + k^2)\psi = 0$, show that the operator $A = \nabla^2 + k^2$ is self-adjoint. Hence, prove the reciprocity of the free space Green's function.

4. (a) A time-harmonic scalar wave $\psi e^{-i\omega t}$ is a superposition of plane waves propagating with wavenumber k at small angles to the horizontal in a uniform 2-dimensional medium (x, z), where x is the horizontal and z the vertical coordinate.

   Derive the parabolic wave equation for the slowly-varying part E of ψ.

   (b) Suppose that a horizontally propagating plane wave encounters a thin vertical layer of thickness δ at $x = -\delta$, which imposes a random phase factor $e^{i\phi(z)}$ on the wave, where ϕ is normally distributed, statistically stationary with respect to z, and has variance σ^2.

   (i) Find the first moment of the Fourier Transform of the field E, $< \hat{E}(x, \nu) >$, as a function of x, for $x \geq 0$.

   (ii) Hence, show that the mean field $< E(x, z) >$ and the mean intensity $< I(x, z) >=< |E(x, z)|^2 >$ are constant with respect to distance x.

   (c) Suppose now that the phase imposed by the layer is a deterministic function $\phi = \cos z$. Find an approximate expression to second order in x for the intensity $I(x, z) = |E(x, z)|^2$ at a small distance x from the screen.

5. Consider a time-harmonic acoustic wave ψ with wavenumber k, normally incident on the plane at $x = 0$, propagating in a random medium with refractive index
$$n \ = \ 1 + \mu W(x, y, z) \ , \quad \text{with} \ \ W = 0 \text{ for } \ x < 0 \ , \tag{1}$$
   where μ is a constant, and W is normally distributed and stationary in y and z, with $< W >= 0$ and $< W^2 >= 1$.

   Assuming that propagation is mainly in the forward direction, at a small angle to the horizontal x, the parabolic equation holds for the reduced wave $E(x, y, z)$. Use the parabolic equation for this case to write the solution for the second moment of the field
$$m_2(x) =< E(x, y_1, z_1)E^*(x, y_2, z_2) >$$
   at an arbitrary point.

Assuming now that the medium is δ-correlated in the direction of propagation x:

$$< W(x, y_1, z_1)W(x, y_2, z_2) >= \delta(x_1 - x_2) < W(y_1, z_1)W(y_2, z_2) > \quad , \tag{2}$$

and isotropic in the (y, z) plane, express this solution in terms of the power spectrum of the medium.
(*You will need to use the following result:*
If

$$F(\nu_\eta, \nu_\zeta) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\eta, \zeta) e^{-i(\nu_\eta \eta + \nu_\zeta \zeta)} d\eta, d\zeta \tag{3}$$

and the kernel is isotropic, i.e. $f(r, \theta) = f(r)$ *in polar coordinates* (r, θ) *the following applies:*

$$F(\nu_\eta, \nu_\zeta) = F(\nu) = \int_0^\infty f(r) J_0(\nu r) r dr \ , \quad \textit{where} \ \ \nu = |(\nu_\eta, \nu_\zeta)| \ , \tag{4}$$

together with the inverse transform

$$f(\nu) = \int_0^\infty F(\nu) J_0(\nu r) \nu d\nu \ . \tag{5}$$

Use also $J_0(0) = 1$.)

6. Consider a linear operator $A : \mathbb{R}^n \mapsto \mathbb{R}^m$, so that A is a real $m \times n$ matrix, which can be factorised using SVD into: $A = U\Sigma V^T$.

   (a) Briefly state the properties of the matrices V, Σ, U^T, and write a generic vector $y \in \mathcal{R}(A)$ as an expansion in an orthonormal basis, using eigenvectors of an appropriate matrix among V, Σ, U^T.

   (b) Given the Moore-Penrose generalised inverse $A^\dagger$ for the problem $Ax = y$: $A^\dagger = U\Sigma^{-1}V^T$,

   (i) show that the generalised inverse solution $x^\dagger$, defined by $x^\dagger = A^\dagger y$ is a solution to the normal equation $A^T Ax = A^T y$;

   (ii) show that the generalised inverse solution $x^\dagger$ is a minimum norm solution.

7. Consider the problem $Ax = y$, where A is an $n \times n$ real, symmetric, positive definite matrix, so A has n (non necessarily distinct) eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$ and corresponding eigenvectors u_i. Assume we are given noisy data $y^{(\delta)}$ such that $\| y^{(\delta)} - y \| \leq \delta$.

   Define a regularising operator by $R_\alpha = \alpha I + A$, where I is the identity and $\alpha > 0$.

   (a) Using the regularising operator R_α, write the regularised solution x_α of the problem with non-noisy data y as an expansion in the basis of eigenvectors u_i with appropriate coefficients.

   (b) Now give an upper bound estimate for the error $\| x - x_\alpha^{(\delta)} \|$ between the exact solution and the regularised solution with noisy data, showing explicit dependence on the measurement error δ and the smallest eigenvalue λ_1 of A.

8. Consider the inverse problem of finding the refractive index $n(\mathbf{r})$ of an inhomogeneity, given an incident field $\psi_{inc}(\mathbf{r})$ generated by a point source ar $\mathbf{r}_0$, and a measured scattered field $\psi_s(\mathbf{r})$.

   (a) write an expression for $\psi_s(\mathbf{r})$ to first order in the distorted wave Born approximation.

   (b) discuss some reasons why this inverse problem is ill-posed.