Part III - Convection
Lent Term 2017

Examples 2

1. Verify the result given in lectures that steady roll-type solutions $Re^{i\ell X}$ to the Newell-Whitehead-Segel equation for $A(X,Y,T)$:

$$A_T = \mu A - |A|^2 A + \left(\frac{\partial}{\partial X} + \frac{1}{2i}\frac{\partial^2}{\partial Y^2}\right)^2 A$$

are unstable to sufficiently long wavelength disturbances when $\ell < 0$.

2. (a) Consider the equation derived in lectures for competing amplitudes on the hexagonal lattice, in the case $b > a > 0$. Derive the stability conditions for the hexagon and roll solution branches that were given in lectures.
(b) now consider the *Boussinesq* problem on the hexagonal lattice, so that there is an extra symmetry $A_i \to -A_i$, and only terms of odd order appear in the equations. Write down all the terms in the evolution equations, consistent with the symmetries, up to fifth order in the amplitude. Show that the behaviour of the phase function Φ is determined by the fifth-order terms, and find the equilibrium values of Φ in that case. What patterns do any new equilibrium values of Φ correspond to when the $|A_i|$ are all equal?

3. Consider the coupled evolution equations $\dot{A} = AF(|A|^2, |B|^2)$, $\dot{B} = BF(|B|^2, |A|^2)$, where $F(X,Y)$ is a polynomial with real coefficients. Let A, B be real without loss of generality, and consider the case $whereb = a + \lambda\epsilon^2$, so that the equations become, correct to 5th order,

$$\dot{A} = \mu A - a(A^2+B^2)A + \epsilon^2(\lambda B^2 A - cA^5 - dA^3B^2 - eAB^4); \quad \dot{b} = \mu B - a(B^2+A^2)B + \epsilon^2(\lambda A^2 B - cB^5 - dB^3A^2 - eB$$

where ϵ is small. Seek solutions with A, B nonzero, and $A^2 \neq B^2$. Show that such solutions can only exist if $A^2 + B^2 = \lambda/(c-e)$. Also show that unless further conditions are satisfied there are in fact no such solutions. Without detailed calculation discuss the effect of adding 7th order terms to the equations.

4. Consider the nonlinear equation for convection between poorly conducting boundaries

$$\frac{\partial \Theta}{\partial t} = -\alpha\Theta - \mu\nabla^2\Theta - \nabla^4\Theta + \nabla\cdot(|\nabla\Theta|^2\nabla\Theta).$$

Show by considering the functional

$$\mathcal{V}[\Theta] = \langle -\alpha|\Theta|^2 + \mu|\nabla\Theta|^2 - |\nabla^2\Theta|^2 - \frac{1}{2}|\nabla\Theta|^4\rangle$$

that the dynamical is variational, with $\mathcal{V}$ increasing monotonically. Show also that $\mathcal{V}$ is bounded above. What do you conclude about the possibility of oscillatory solutions?
Prove also that any steady solution of the system has zero average.
Suppose there is a spatially periodic roll-type steady solution with $\Theta = F(x)$. Show that $\mathcal{V}[F(x) + \delta F(y)] > \mathcal{V}[F(x)]$ for sufficiently small δ. What you deduce about the stability of roll solutions?

5. The amplitude of slowly modulated hexagonal convection can be modelled by the real equation for $A(x, y, t)$:

$$\dot{A} = \mu A + \alpha A^2 - cA^3 + \nabla^2 A.$$

Consider the situation $\mu < 0 < \alpha, c$. In what range of $\mu < 0$ can there be two steady stable uniform solutions (i.e. A independent of x, y, t), one $A = 0$ and the other $A = A_0 > 0$?
Now suppose that μ is in this range and look for 'front' solutions with $A = F(\xi)$, $\xi = x - Vt$, $A \to 0$, $x \to \infty$, $A \to A_0$, $x \to -\infty$. Without solving the equation for $F(\xi)$ show that the sign of V depends on whether μ is greater than or less than a critical value μ_c.

6. The equation governing the onset of convection in some complicated problems (for example convection in the presence of solidification) can take the generalised Landau-Ginzburg form for the complex amplitude $A(X, t)$

$$\dot{A} = \mathcal{L}[A] - |A|^2 A,$$

where the linear differential operator $\mathcal{L}$ is defined by $\mathcal{L}[e^{ikX}] = f(k)e^{ikX}$, and $f(k)$ is a real function of k.
Show that the condition for marginal stability to long-wavelength disturbances of the steady solution $A = R(k)e^{ikX}$, $R^2 = f(k)$ is $f(k)f''(k) + (f'(k))^2 = 0$. Verify that this condition gives the correct answer for the usual convection case where $f(k) = \mu - k^2$.

7. Consider the model equation

$$\dot{\psi} = \mu\psi - (1 + \frac{\partial^2}{\partial x^2})^2\psi + \alpha\psi^2 - \psi^3$$

for a function $\psi = \psi(x, t)$. (This is the (modified) *Swift-Hohenberg equation*).
Show that there are marginal solutions proportional to $e^{i\mathbf{k}\cdot\mathbf{x}}$ with $|\mathbf{k}| = \mathbf{1}$ when $\mu = 0$
Consider two-dimensional small amplitude solutions of the form $\psi = \epsilon(Ae^{ix} + c.c.) + \ldots$ for small μ. Show that in spite of the quadratic term in the equation the complex amplitude A obeys an equation of the form $\dot{A} = \mu A - \lambda|A|^2 A$ (i.e. with no quadratic terms). Explain why this must be so on grounds of symmetry. Determine λ and show it becomes negative when α is sufficiently large.