Mathematical Tripos Part III 2007–08
Fundamentals of Atmosphere–Ocean Dynamics

Examples 2

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In these examples, 'BN' refers to the background lecture notes for the course.

1. [From the 1997-8 examination on this course] Write down the Boussinesq momentum, mass-continuity and buoyancy equations for a continuously stratified, non-rotating, ideal fluid. Buoyancy effects should be represented in the equations by a suitably defined buoyancy acceleration $\sigma(\mathbf{x}, t)$, to be specified. State briefly the conditions under which the Boussinesq approximation is valid, given that the fluid is incompressible.
   For 2-dimensional motion with $\partial/\partial y = 0$, derive the y-component of the corresponding vorticity equation in the form
   $$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial\sigma}{\partial x} = \frac{\partial\psi}{\partial x}\frac{\partial}{\partial z}\nabla^2\psi - \frac{\partial\psi}{\partial z}\frac{\partial}{\partial x}\nabla^2\psi ,$$
   where $\psi(x, z, t)$ is a streamfunction to be specified.
   For small disturbances about rest, derive the dispersion relation, and the formula for group velocity, for a 2-dimensional sinusoidal plane-wave small disturbance, with disturbance quantities all of the form $\exp(ikx + imz - i\omega t)$ times a complex amplitude. Comment briefly on the fact that frequency is independent of wavelength and on its implication regarding the direction of the group velocity. Show that all such plane waves are finite-amplitude solutions, i.e., that they satisfy the equations before linearization. Show further that certain 2-dimensional superpositions of plane waves, to be specified, are finite-amplitude solutions. Comment briefly on the implications for models of stratified flow over topography.

2. Verify that the shallow water equations, linearized about rest in an inertial frame of reference, admit a solution corresponding to the problem of BN pp. I.37–40, provided that the applied force field $F = f(x, t)$ is independent of z. (Why is this independence of z required?) Derive the equation corresponding to BN eq.(I.2.26b) and hence write down expressions for horizontal velocity u and layer depth h, corresponding to the expressions BN (I.2.27) in a manner to be specified. In particular, how is the positive constant c related to layer depth?

   Describe, with the help of suitable sketches, what fluid motions these solutions represent when $f(x, t)$ has the same general character as that chosen for illustrative purposes on p.I.37.

   Consider now the case $f(x, t) = \delta(x)\phi(t)$, where ϕ is an arbitrary real-valued function of t. Show in two ways – first from the counterparts to the expressions (I.2.27), and second by solving the problem afresh – that all disturbance quantities in the region $x > 0$ are proportional to $\phi(t - x/c)$, and in $x < 0$ proportional to $\phi(t + x/c)$. Find the constants of proportionality. Show further that the same solution, apart from differing constants of proportionality, describes the response in $x > 0$ to the small-amplitude motion of a piston near $x = 0$, having a prescribed velocity $\propto \phi(t)$.

3 **Optional:** Run the QBO simulation from the GEFD Summer School computer demonstrations. It is based on the group-velocity theory. [In a Command window, after logging on to a PWF windows workstation, type `I:\CATAM\GEFD` then carriage return; then select the QBO demonstration with the arrow keys and another carriage return. You may want to increase the numerical resolution if it runs too fast. When finished, you can exit from everything using the ESCAPE key (twice).] Use the default parameter values but draw in different initial $\bar{u}$ profiles with the mouse, to show that

the establishment of the QBO is robust in the sense of being insensitive to initial conditions; $\bar{u}(z,t)$ is shown on the left of the display, and the wave-induced momentum fluxes $\overline{u'w'}$ on the right.

4. Internal gravity waves generated by flow past an island are trapped in a waveguide by the earth's surface and by the effects of a z-dependent mean flow $\{\bar{u}(z), \bar{v}(z), 0\}$ and a z-dependent buoyancy frequency $N(z)$. Neglecting rotation, show that the linearized Boussinesq equations admit stationary-wave solutions $\propto e^{i(kx+ly)}$ $(\partial/\partial t = 0)$, with z-dependent complex amplitudes, provided that the complex amplitude $\widehat{w}(z)$ for the vertical disturbance velocity w' satisfies the second-order ordinary differential equation

$$\widehat{w}_{zz} + \{L^2(z) - k^2 - l^2\}\widehat{w} = 0 \ ,$$

where

$$L^2(z) = \frac{N^2(z)}{U^2(z)} - \frac{U_{zz}(z)}{U(z)} \ , \quad \text{and} \ \ U(z) = \frac{k\bar{u} + l\bar{v}}{(k^2+l^2)^{\frac{1}{2}}} \ .$$

[Hint: a shortcut is to rotate the xy axes after considering first the case $l = 0$, which reduces to the two-dimensional theory in BN p.82ff. (Why?)] What restoring mechanisms are represented by the two terms in L^2? If U increases with altitude z, while N^2 is roughly constant and $\gg UU_{zz}$, discuss, without solving the equation, why waveguide modes should exist.

Show further that, if $\widehat{w} = 0$ at each end of the range of integration,

$$k^2 + l^2 = I(\widehat{w}(\cdot)) \equiv \frac{\int (L^2\widehat{w}^2 - \widehat{w}_z^2)dz}{\int \widehat{w}^2 dz} \ .$$

Now using the fact that a small change in the function $\widehat{w}(z)$ produces only a second-order small change in the functional $I(\widehat{w}(\cdot))$ [Rayleigh quotient stationarity property: do not prove this], deduce that the horizontal group velocity c_{g} relative to the earth's surface has magnitude

$$|c_{\mathrm{g}}| = \left| \frac{2(k^2+l^2)\int \widehat{w}^2 dz}{\int \{U^{-3}N^2 + U^{-1}L^2\}\widehat{w}^2 dz} \right|$$

and that $\operatorname{sgn} c_{\mathrm{g}} = \operatorname{sgn} U$.

[This is the essential explanation for the 'ship-wake' lee-wave patterns in the satellite pictures shown in lectures, although the U_{zz} restoring mechanism reinforces that from N to some extent in the real case. See BN p.84ff. for the solution to this problem and for its use in explaining observed mountain-wave patterns seen in satellite photographs.]

Please email any corrections/comments to mem@damtp.cam.ac.uk