Nonlinear Continuum Mechanics
Problem Sheet 1

1. Given $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$ and $\mathbf{F}^T\mathbf{F}\mathbf{N}_r = \lambda_r^2\mathbf{N}_r$ (in the usual notation), show that

(a) $\mathbf{U}\mathbf{N}_r = \lambda_r\mathbf{N}_r$, (b) If $\mathbf{n}_r = \mathbf{R}\mathbf{N}_r$, $\mathbf{V}\mathbf{n}_r = \lambda_r\mathbf{n}_r$,

(c) $\mathbf{F}\mathbf{N}_r = \lambda_r\mathbf{n}_r$, (d) $\mathbf{F}^T\mathbf{n}_r = \lambda_r\mathbf{N}_r$.

2. For the simple shear

$$\mathbf{F} = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$$

find $\mathbf{R}$, $\mathbf{U}$ and $\mathbf{V}$ by verifying the identity

$$(1 + \tfrac{1}{4}\gamma^2)\begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}\gamma \\ -\frac{1}{2}\gamma & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{2}\gamma \\ \frac{1}{2}\gamma & 1 + \frac{1}{2}\gamma^2 \end{pmatrix}$$

and then rearranging. Find the principal axes of $\mathbf{U}$ and the corresponding principal stretches.

3. Under a uniform deformation $\mathbf{X} \to \mathbf{x} = \mathbf{FX}$, let

$$\mathbf{X}_1 \to \mathbf{x}_1 = \mathbf{F}\mathbf{X}_1, \quad \mathbf{X}_2 \to \mathbf{x}_2 = \mathbf{F}\mathbf{X}_2$$

and assume that $\mathbf{X}_1$, $\mathbf{X}_2$ are orthogonal unit vectors. If $\mathbf{X}_3 = \mathbf{X}_1 \wedge \mathbf{X}_2$, show that $\mathbf{F}^T(\mathbf{x}_1 \wedge \mathbf{x}_2) = J\mathbf{X}_3$. Reconcile this with Nanson's formula [let $\{\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3\}$ be a basis and use $J = \varepsilon_{ijk}F_{i1}F_{j2}F_{k3}$].

4. Show that the nominal stress

$$\mathbf{P} = (P_{Ii}) = \begin{pmatrix} -X_1X_2/R^3 & X_2X_3/R^3 & 0 \\ -1/R + X_1^2/R^3 & 0 & -2X_1^2X_3/R^4 \\ 0 & -X_1X_2/R^3 & 2X_1^2X_2/R^4 \end{pmatrix}$$

is in equilibrium without body-force, and find the components of nominal traction on the surface $R = a$. [$R = (X_1^2 + X_2^2 + X_3^2)^{1/2}$].

5. Let $\mathbf{e}_I = F^i_{.I}\mathbf{e}_i$, where $F^i_{.I} = \partial x^i/\partial X^I$. Show that

$$\frac{\partial \mathbf{e}_I}{\partial X^J} = \left\{ \begin{matrix} K \\ IJ \end{matrix} \right\} \mathbf{e}_K,$$

where

$$\left\{ \begin{matrix} K \\ IJ \end{matrix} \right\} = \tfrac{1}{2}g^{KL}[g_{IL,J} + g_{JL,I} - g_{IJ,L}],$$

with $g_{IJ} = F^i_{.I}F^j_{.J}g_{ij}$, $g^{IK}g_{KJ} = \delta^I_J$ and $g_{ij} = \delta_{ij}$ [coordinates x^i are Cartesian, $_{,K}$ means $\partial/\partial X^K$]. Deduce the formula for $\partial\mathbf{V}/\partial X^J$, where the vector $\mathbf{V}$ is given as $\mathbf{V} = V^i\mathbf{e}_i = V^I\mathbf{e}_I$.

6. Prove that $\partial J/\partial F_{Ii} = \frac{1}{2}\varepsilon_{ijk}\varepsilon_{IJK}F_{jJ}F_{kK}$. Use this result, in conjunction with $\boldsymbol{\sigma} = J^{-1}\mathbf{FP}$, to obtain the equations of motion in Lagrangian form, from the corresponding Eulerian equations.

7. A stress-rate (taken with the current state as reference) is called "objective" if, under a rotation $(\mathbf{Q}(t))^T$ of the frame of reference, so that $\boldsymbol{\sigma} \to \boldsymbol{\sigma}' = \mathbf{Q}\boldsymbol{\sigma}\mathbf{Q}^T$, the considered stress-rate transforms in the same way. Verify directly that the "contravariant Kirchhoff" rate $\delta\sigma_{ij}/\delta t$, and the Jaumann rate $\mathcal{D}\sigma_{ij}/\mathcal{D}t$, are objective.

8. Adopting the polar decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$, show that the spin $\mathbf{W}$ can be expressed

$$\mathbf{W} = \boldsymbol{\Omega} + \tfrac{1}{2}\mathbf{R}(\dot{\mathbf{U}}\mathbf{U}^{-1} - \mathbf{U}^{-1}\dot{\mathbf{U}})\mathbf{R}^T,$$

where $\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^T$. Prove that the stress-rate

$$\left(\frac{\mathcal{D}'}{\mathcal{D}'t}\right)\boldsymbol{\sigma} = \dot{\boldsymbol{\sigma}} - \boldsymbol{\Omega}\boldsymbol{\sigma} + \boldsymbol{\sigma}\boldsymbol{\Omega}$$

is objective.

9*. The orthonormal set of vectors $\{\mathbf{N}_K\}$, defined so that

$$\mathbf{F}^T\mathbf{F}\mathbf{N}_K = \lambda_K^2\mathbf{N}_K$$

is called "the Lagrangian triad". It is obtained from the reference triad $\{\mathbf{e}_K^0\}$ by a rotation $\mathbf{R}^L$:

$$\mathbf{R}^L = \sum_{K=1}^{3} \mathbf{e}_K^0 \otimes \mathbf{N}_K = \sum_{K=1}^{3} \mathbf{e}_K^0 \otimes R_{KL}^L\mathbf{e}_L^0.$$

In matrix representation with $\{\mathbf{e}_K^0\}$ as basis,

$$(\mathbf{R}^L) = (\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3) \equiv (R_{IJ}^L); \quad R_{IJ}^L = \mathbf{e}_I^0 \cdot \mathbf{N}_J$$

and

$$\mathbf{F}^T\mathbf{F} = \sum_{K=1}^{3} \lambda_K^2\mathbf{N}_K\mathbf{N}_K^T.$$

Assuming that the eigenvalues λ_K are distinct, show that the spin $\boldsymbol{\Omega}^L$ of the Lagrangian triad has matrix representation with $\{\mathbf{N}_K\}$ as basis,

$$(\Omega_{KM}^L) = \begin{cases} \frac{1}{\lambda_K^2 - \lambda_M^2}\mathbf{N}_K^T(\dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}})\mathbf{N}_M, & K \neq M \\ 0, \quad K = M. \end{cases}$$