Nonlinear Continuum Mechanics, Problem Sheet 3

1. A Green-elastic material has stored energy function W per unit referece volume and is subject to two constraints:

$$\phi_1(\mathbf{F}) = 0 \quad \text{and} \quad \phi_2(\mathbf{F}) = 0.$$

Show that the relation between nominal stress and deformation gradient for this material is

$$P_{Ii} = \frac{\partial W}{\partial F_{iI}} + q_1 \frac{\partial \phi_1}{\partial F_{iI}} + q_2 \frac{\partial \phi_1}{\partial F_{iI}},$$

where q_1 and q_2 are undetermined.

A body comprising such material occupies a domain Ω in its reference configuration and is in equilibrium in the presence of dead-load body force and some combination of dead-load traction and placement ($\mathbf{x}$ prescribed) conditions on its boundary $\partial\Omega$. Show that its equilibrium configuration is stable, and the solution branch is incrementally unique, if

$$\int_\Omega \frac{\partial u_i}{\partial X_I} \left[\frac{\partial^2 W(\mathbf{F})}{\partial F_{iI} \partial F_{jJ}} + q_1 \frac{\partial^2 \phi_1(\mathbf{F})}{\partial F_{iI} \partial F_{jJ}} + q_2 \frac{\partial^2 \phi_2(\mathbf{F})}{\partial F_{iI} \partial F_{jJ}} \right] \frac{\partial u_j}{\partial X_J} \, \mathrm{d}\mathbf{X} > 0$$

for all non-zero perturbations $\delta\mathbf{x} = \lambda\mathbf{u}$ ($\lambda \ll 1$) compatible with the constraints and any placement boundary conditions. Here, $\mathbf{F}$, q_1 and q_2 are the actual equilibrium fields.

2. Show that the "upper convected Maxwell" constitutive relation for incompressible fluid,

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^d - p\mathbf{I}; \quad \frac{\delta \boldsymbol{\sigma}^d}{\delta t} + \frac{\boldsymbol{\sigma}^d}{\tau} = \frac{2\mu}{\tau}\mathbf{D},$$

can be expressed in the integral form

$$\boldsymbol{\sigma}^d(t) = \frac{2\mu}{\tau} \int_{-\infty}^t e^{-(t-t')/\tau} \mathbf{F}(t)\mathbf{F}^{-1}(t')\mathbf{D}(t')\mathbf{F}^{-T}(t')\mathbf{F}^T(t) \, \mathrm{d}t'.$$

[*Recall that the relation* $\boldsymbol{\sigma} = \mathbf{F}\mathbf{T}\mathbf{F}^T$ *between Cauchy and second Piola–Kirchhoff stress for incompressible material equivalently interprets the components of* $\mathbf{T}$ *as contravariant components of* $\boldsymbol{\sigma}$ *on convected coordinates.*]
Check directly that the integral form satisfies the principle of material indifference.

Develop corresponding relations (differential and integral) for the "lower convected Maxwell" model. [*Recall:* $\sigma_{IJ} = F^i_I F^j_J \sigma_{ij}$ *defines the required covariant components of* $\boldsymbol{\sigma}$.]

3. Obtain $\boldsymbol{\sigma}^d(t)$ explicitly for the two models discussed in question 2, in the case of a time-dependent simple shear:

$$\mathbf{F}(t) = \begin{pmatrix} 1 & \gamma(t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

4. Develop the solution for laminar flow between two plates in the presence of a pressure gradient, for "upper convected Maxwell" fluid. [*Use the integral form of the constitutive relation.*]

Now make the problem non-trivial by considering a *time-dependent* pressure gradient, in the "slow flow" limit so that inertia is disregarded. If $\dot\gamma(x_2, t) = \partial v(x_2, t)/\partial x_2$, show that $\gamma(x_2, t)$ must satisfy

$$\frac{1}{\tau} \int_{-\infty}^t e^{-(t-t\prime)/\tau} \dot\gamma(x_2, t') \mathrm{d}t' = -\left[\frac{G(t)}{\mu} + A(t)\right],$$

where $G(t)$ is the pressure gradient and $A(t)$ is an arbitrary function. Deduce $v(x_2, t)$.

5. Solve the problem of Couette flow, in the region $a < r < b$, $-\infty < z < \infty$, with the boundary conditions $v_\theta(a) = v_0$, (constant) $v_\theta(b) = 0$, for "upper convected Maxwell" fluid. [*Try to avoid a lot of work by using the integral form of the constitutive relation and a suitable rotating frame of reference.*]