Nonlinear Continuum Mechanics, Problem Sheet 4

1. Slow flow of "generalised Newtonian" fluid satifies the equations

$$\mathrm{div}\boldsymbol{\sigma} = 0, \;\; \mathrm{div}\mathbf{v} = 0, \;\; \sigma_{ij} = -p\delta_{ij} + \mu((\|\mathbf{D}\|)D_{ij} \;\text{ in a domain } \mathcal{D},$$

where $\|\mathbf{D}\| = (D_{ij}D_{ij})^{1/2}$, together with $\mathbf{v} = \mathbf{v}^0$, given, on the boundary $\partial\mathcal{D}$ of the domain. Show that the constitutive relation can be written

$$\sigma_{ij} = -p\delta_{ij} + \frac{\partial\Omega}{\partial D_{ij}}, \;\; \Omega(\|\mathbf{D}\|) = \int_0^{\|\mathbf{D}\|} s\mu(s)\mathrm{d}s.$$

Assuming that $\mu(0) > 0$ and $(s\mu(s))' > 0$ for all $s \geq 0$, show that the actual flow minimises the integral

$$\int_{\mathcal{D}} \Omega(\|\mathbf{D}\|)\mathrm{d}\mathbf{x},$$

amongst all incompressible flows $\mathbf{v}'$ for which $\mathbf{v}' = \mathbf{v}^0$ on $\mathcal{D}$. [*Show that*

$$\Omega(\|\mathbf{D}'\|) - \Omega(\|\mathbf{D}\|) - \sigma_{ij}(D'_{ij} - D_{ij}) \geq 0,$$

i.e., that Ω *is a convex function; then use the fact that* $\mathbf{v}$ *satisfies all of the conditions above.*]

2*. Let $\Omega^*(\boldsymbol{\sigma})$ be the potential dual to Ω: $\Omega^*(\boldsymbol{\sigma}) = \sup_{\mathbf{D}} \{\sigma_{ij}D_{ij} - \Omega(\|\mathbf{D}\|)\}$. Show that the actual stress minimises the integral $\int_{\mathcal{D}} \Omega^*(\boldsymbol{\sigma})\,\mathrm{d}\mathbf{x}$, amongst all divergence-free stress fields. [*Show that* $\Omega^*(\boldsymbol{\sigma}') - \Omega(\boldsymbol{\sigma}) - (\sigma'_{ij} - \sigma_{ij})D_{ij} \geq 0$, *i.e., that* Ω^* *is a convex function.*]

3. Show that equation $(21.11)_2$ of the notes implies that

$$\mathrm{d}T_{IJ} = \left\{\Lambda_{IJKL} - \frac{[\Lambda_{IJMN}(\partial f/\partial T_{MN})][\Lambda_{KLRS}(\partial f/\partial T_{RS})]}{h + (\partial f/\partial T_{MN})\Lambda_{MNRS}(\partial f/\partial T_{RS})}\right\}\mathrm{d}E_{KL}$$

during loading. Recall that f and g are related so that $f(\mathbf{T}) = 0$ and $g(\mathbf{E}) = 0$ when $\mathbf{T}$ and $\mathbf{E}$ represent stress and strain at the same state of yield; show that $\Lambda_{IJKL}(\partial f/\partial T_{KL})$ must be co-directional with $(\partial g/\partial E_{IJ})$.[*Consider elastic increments.*]

4. (a) For the stress $\boldsymbol{\sigma} = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_2)$, show that $\sigma'_{ij}\sigma'_{ij} = k^2$ implies that $|\sigma_1 - \sigma_2| = (3/2)^{1/2}k$.
(b) A hollow sphere of incompressible Von Mises material has, before deformation, inner radius a_0 and outer radius b_0. Pressure on the inner boundary is gradually increased, while the outer boundary remains traction-free. Find the pressure P_0 at which the cavity first begins to expand. [*Assume a stress field, in polar coordinates, of the form given in part (a), with* σ_1 *in the radial direction. The radial equation of equilibrium in polar coordinates is* $\mathrm{d}\sigma_{rr}/\mathrm{d}r + 2(\sigma_{rr} - \sigma_{\theta\theta})/r = 0$.]
(c) Suppose now that the internal pressure $P(t)$ is applied, for $t \geq 0$, with $P(0) > P_0$. By considering balance of work-rate or otherwise, show that $a(t)$ satisfies the differential equation

$$\sqrt{6}k\ln(b/a) + \tfrac{1}{2}\rho\dot{a}^2\left(3 - 4a/b + a^4/b^4\right) + \rho\ddot{a}\left(a - a^2/b\right) = P(t),$$

where $b^3 = a^3 + b_0^3 - a_0^3$ and ρ denotes the uniform mass density of the material.

5. (a) In "longitudinal shear" deformation, the only non-zero component of velocity is $v_3(x_1, x_2)$ and the only non-zero stress components are $\sigma_{13}(x_1, x_2)$ and $\sigma_{23}(x_1, x_2)$. The Von Mises yield criterion reduces, under such deformation, to $\sigma_{13}^2 + \sigma_{23}^2 = \sigma_Y^2$; it is satisfied identically by taking

$$\sigma_{13} = -\sigma_Y\sin\phi, \quad \sigma_{23} = \sigma_Y\cos\phi.$$

Show that ϕ is constant on any line along which $\mathrm{d}x_2/\mathrm{d}x_1 = \tan\phi$. [*Show that* $(\mathrm{d}/\mathrm{d}s)\phi(x_1(s), x_2(s)) = 0$ *on any such line. Such lines form the single family of characteristics.*]

(b) A crack in Von Mises material occupies a segment of the x_1-axis and its right-hand end is at the origin. If the body is in a state of longitudinal shear, symmetric about the x_1-axis, show that a possible pattern of characteristics close to the crack tip is a fan, centred at the origin, and extending over $-\pi/2 < \phi < \pi/2$. Deduce that, relative to cylindrical polar coordinates (r, ϕ, x_3)), the only non-zero stress component is $\sigma_{\phi 3} = \sigma_Y$; assuming also the associated flow law, show that v_3 is a function of ϕ only. [*The state of stress to the left of the centred fan region is a uniform stress* σ_{13}*, so the crack surfaces are traction-free. In fact, this solution is asymptotic; the stress there will be elastic, and will fall below yield as distance from the crack tip increases.*]

(c)* The purely elastic solution for this crack has the form near the crack tip

$$\sigma_{13} = -\frac{K}{\sqrt{2\pi r}}\sin(\phi/2), \quad \sigma_{23} = \frac{K}{\sqrt{2\pi r}}\cos(\phi/2), \quad u_3 = \sqrt{\frac{2}{\pi}}\frac{K}{\mu}\sqrt{r}\sin(\phi/2).$$

Show that elastic and plastic solutions can be put together by adopting the field constructed in part (b) in a circular region, of radius a, centred a distance a ahead of the crack, with an elastic field, of the form given above, just outside it, except that the elastic field is defined by polar coordinates (r', θ), centred at the centre of the circle. Find the radius a in terms of the "stress intensity factor" K. [*The elastic field leaves the crack faces traction-free. Stresses and displacement must be continuous across the elastic-plastic boundary.*]

6. In terms of the multiplicative decomposition $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$, show that $\mathbf{L} = \mathbf{L}^e + \mathbf{L}^p$, where $\mathbf{L}^e = \dot{\mathbf{F}}^e(\mathbf{F}^e)^{-1}$ and $\mathbf{L}^p = \mathbf{F}^e\{\dot{\mathbf{F}}^p(\mathbf{F}^p)^{-1}\}(\mathbf{F}^e)^{-1}$. Show that

$$\mathbf{A} = (\mathbf{F}^p)^{-1}(\mathbf{F}^e)^{-1}\boldsymbol{\tau}\mathbf{F}^e$$

(where $\mathbf{A}$ is as defined in the notes and $\boldsymbol{\tau} = \{\tau_{ji}\}$ is Kirchhoff stress) and check that

$$\tau_{ji}D^p_{ij} \equiv \tau_{ji}L^p_{ij} = A_{JI}\dot{F}^p_{IJ} = T^p_{JI}\dot{F}^p_{IK}\{(\mathbf{F}^p)^{-1}\}_{KJ},$$

where $\mathbf{D}^p = \frac{1}{2}[\mathbf{L}^p + (\mathbf{L}^p)^T]$ and $\mathbf{T}^p = (\mathbf{F}^e)^{-1}\boldsymbol{\tau}\mathbf{F}^e$. [*Thus,* $\mathbf{T}^p$ *has an interpretation in terms of mixed components of Kirchhoff stress, on a frame deformed from the "intermediate" configuration.*]