A star $*$ denotes a question, or part of a question, that should not be done at the expense of unstarred questions. Corrections, suggestions and comments should be emailed to `l.j.ayton@maths.cam.ac.uk`.

1. Find the appropriate scalings for each root of

(a)
$$\epsilon x^4 - x^2 - x + 2 = 0\,,$$

(b)
$$\epsilon^2 x^3 + x^2 + 2x + \epsilon \;=\; 0\,.$$

Hence find two terms in the approximation for each root.

2. Find an asymptotic approximation to the exponential integral
$$E_n(x) \quad = \quad \int_1^\infty t^{-n} e^{-xt}\, dt\ ,$$
for real $n = \text{ord}(1)$ and $x \to \infty$, and estimate the remainder. How big is the remainder for the best choice of the number of terms in the expansion?

3. Evaluate the first two terms as $r \to 0$, and the first two terms as $r \to \infty$, of
$$\int_0^\infty \frac{rx\,dx}{(r^2+x)^{3/2}(1+x)}\ .$$

$*$4. The integral $I(\lambda)$ is defined by
$$I(\lambda) = \int_0^\infty s \exp\left(-s^2 + 2\lambda - \frac{\lambda^2}{s^2}\right) \mathrm{d}s\,.$$
Find the asymptotic expansion for $I(\lambda)$ as $\lambda \to 0$ correct to, and including, terms that are $\mathcal{O}(\lambda^2)$. It may help to recall that
$$\int_0^\infty \ln(t) \exp(-t)\, \mathrm{d}t = -\gamma\,,$$
where γ is Euler's constant.

5. For real x and $x \to \infty$, find the full asymptotic behaviour of
$$K_0(x) \quad = \quad \int_1^\infty (t^2-1)^{-\frac{1}{2}} e^{-xt}\, dt\ .$$

6. For real x and $x \to \infty$, find the leading-order asymptotic behaviour of

(a)
$$\int_0^1 \sin(t)\, e^{-x \sinh^4 t}\, dt\ ,$$

(b)
$$\int_0^\infty e^{-xt - t^{-1}}\, dt\ .$$

7. For real n and $n \to \infty$, find the leading-order asymptotic behaviour of

$$J_n(n) \quad = \quad \frac{1}{\pi}\int_0^{\pi} \cos(n \sin t - nt)\, dt \ .$$

[Give your answer in a form that is explicitly real. It should involve $\Gamma(1/3)$.]

8. Find the asymptotic behaviour of

$$J_\nu(\nu \operatorname{sech}\alpha) \quad = \quad \frac{1}{2\pi \mathrm{i}}\int_{\infty - \mathrm{i}\pi}^{\infty + \mathrm{i}\pi} e^{\nu \operatorname{sech}\alpha \sinh t - \nu t}\, dt \ ,$$

for real ν and α, as $\nu \to \infty$ with first $\alpha > 0$ and second $\alpha = 0$. (There are lots of saddles here. A contour plot using MATLAB may help convince you that you have a steepest descent contour. The case $\alpha = 0$ has a cubic saddle where three ridges meet and $\phi'' = 0$.)

9. The function $f(y;\lambda)$ is defined by

$$f(y;\lambda) = \int_C \exp\left(\lambda(1 + \mathrm{i}y)z - \tfrac{1}{3}z^3\right) dz,$$

where y and λ are real, and the contour C starts from $z = 0$ and extends to $z = \infty$ in the sector $|\arg(z)| < \pi/6$.

(a) Find the leading-order asymptotic behaviour of $f(0;\lambda)$ as $\lambda \to -\infty$.

(b) Find the leading-order asymptotic behaviour of $f(0;\lambda)$ as $\lambda \to +\infty$.

(c) By considering the solutions deduced in parts (a) and (b), and the steepest descent contours, find the leading-order asymptotic behaviour of f for $0 \le y < \infty$. In particular:

   i. state clearly your choice of integration contour;

   ii. for $\lambda \gg 1$ comment on how the asymptotic behaviour of the solution differs according as $0 \le y < y_c$ and $y_c < y < \infty$, where y_c should be identified.

(d) Show that $f(y;\lambda)$ satisfies the differential equation

$$f_{yy} + \lambda^3(1 + \mathrm{i}y)f = -\lambda^2 \ ,$$

with boundary conditions $f \to 0$ as $|y| \to \infty$.

(e) * In relation to this equation, why is it that

$$f = -\frac{1}{\lambda(1 + \mathrm{i}y)} - \frac{2}{\lambda^4(1 + \mathrm{i}y)^4} + \dots \ ,$$

is *not* always a uniformly valid asymptotic approximation for $|\lambda| \gg 1$? * [This issue will be considered in the matched asymptotic expansions section of the course later.]