Classical Wave Scattering
Part III

O. Rath Spivack

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This is just a first draft of the material covered in this course. I should very much appreciate being told of any corrections or possible improvements Comments, please, to O.Rath-Spivack@damtp.cam.ac.uk.

Contents

1 Governing equations for acoustic and electromagnetic waves

1.1 Acoustic Waves

Acoustic waves are mechanical disturbances that propagate in a medium, for example air, water, or a solid structure such as, for example, the shell of a ship.
The governing equations that describe how acoustical waves propagate in a medium are derived from the basic conservation laws for fluids and the laws of thermodynamics. The detailed derivation can be found in many books (see, for example, Pierce [8], LD Landau and EM Lifschitz, *Fluid Mechanics*). Here we shall just give the main steps.
For a fluid with density ρ and velocity $\mathbf{v}$, the **conservation of mass** in the non-relativistic case is expressed as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \tag{1.1}$$

In the presence of a mass source, the r.h.s. will be non-zero. If we have an ideal fluid, i.e. with zero viscosity, the surface force $\mathbf{F}_s$ is directed normally into the surface, so $\mathbf{F}_s = -\mathbf{n}p$, where p is the pressure. With this assumption and neglecting gravity and any other external forces, the **conservation of momentum** is expressed as:

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p \tag{1.2}$$

In general, for a system in local thermodynamic equilibrium, an **equation of state** will hold, whereby a function of state can be expressed in terms of any other two. We shall use the equation of state that relates the pressure to the entropy S and the density ρ of the system: $p = p(\rho, S)$, where the entropy is a measure of the disorder of a system and is such that, for a reversible process, the differential change in entropy equals the differential change in absorbed heat divided by the system temperature: $dS = dQ/T$. If one can ignore any heat flow (so no temperature gradient imposed externally, gradient of the fluid small), then the fluid motion is adiabatic and the entropy is constant, S_0 (*isentropic* process). In this case

$$p = p(\rho, S_0) \tag{1.3}$$

and depends only on the density.

1.1 Acoustic Waves

If sound, i.e. acoustic disturbance, is regarded as a small-amplitude perturbation of an ambient state $(\rho_0, p_0, \mathbf{v}_0)$, then when the disturbance is present one has

$$p = p_0 + p'; \quad \rho = \rho_0 + \rho'; \quad \mathbf{v} = \mathbf{v}_0 + \mathbf{v}',$$

which also satisfy the conservation laws:

$$\frac{\partial(\rho_0 + \rho')}{\partial t} + \nabla \cdot ((\rho_0 + \rho')\mathbf{v}') = 0 \tag{1.4}$$

$$(\rho_0 + \rho') \left(\frac{\partial \mathbf{v}'}{\partial t} + \mathbf{v}' \cdot \nabla \mathbf{v}' \right) = -\nabla(p_0 + p') \tag{1.5}$$

and

$$p_0 + p' = p(\rho_0 + \rho', S_0) \tag{1.6}$$

Here $\mathbf{v}_0 = 0$, so this derivation applies in the absence of mean flow. We shall also take the fluid to be homogeneous so p_0 and ρ_0 are constants related by $p_0 = p(\rho_0, S_0)$.
If p' is expanded in a Taylor series in ρ':

$$p' = \left(\frac{\partial p}{\partial \rho} \right)_0 \rho' + \frac{1}{2} \left(\frac{\partial^2 p}{\partial \rho^2} \right)_0 (\rho')^2 + \dots ,$$

by using this expansion in the above equations and truncating to first order we obtain the ***linear acoustic equations***

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0 \tag{1.7}$$

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' \tag{1.8}$$

$$p' = \left(\frac{\partial p}{\partial \rho} \right)_0 \rho' = c^2 \rho'. \tag{1.9}$$

The quantity c introduced in (1.9) is denoted *speed of sound* for reasons that will become clear once we obtain the solution to the equation governing the time evolution of the acoustic pressure p. The factor of proportionality ρc, equal to the ratio between pressure and velocity, is called ***characteristic impedance*** of the medium. Let's now use (1.9) in (1.7):

$$\frac{1}{c^2} \frac{\partial p'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0 \tag{1.10}$$

If we then take the partial time derivative of (1.10) and use (1.8), we obtain the **wave equation** for the acoustic pressure:

$$\nabla^2 p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0, \tag{1.11}$$

where we have now dropped the primes.
The wave equation can be formulated alternatively in terms of a ***velocity potential***. If we take the curl of (1.8) and use the vector identity $\nabla \times (\nabla \varphi) = 0$, valid $\forall \varphi$, it follows that (again dropping all primes)

$$\frac{\partial(\nabla \times \mathbf{v})}{\partial t} = 0,$$

i.e. the ***vorticity*** $(\nabla \times \mathbf{v})$ is constant in time. Therefore the velocity field is irrotational $(\nabla \times \mathbf{v} = 0)$ if it is irrotational initially, and we can introduce a velocity potential ϕ by writing

$$\mathbf{v} = \nabla \phi. \tag{1.12}$$

Note that $\mathbf{v} = \nabla\phi + \mathbf{v_0}$ will apply if the fluid is initial moving with velocity $\mathbf{v_0}$. Substituting (1.12) in (1.8), we obtain

$$p = \rho_0 \frac{\partial \phi}{\partial t}. \tag{1.13}$$

Now, using (1.12), (1.13) and (1.9) in (1.8) gives

$$\nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0, \tag{1.14}$$

which is the wave equation in terms of the velocity potential.
A general solution of (1.11) is

$$p = f\left(t - \frac{\xi}{c}\right) + g\left(t + \frac{\xi}{c}\right), \tag{1.15}$$

where f and g are arbitrary functions which will be determined by initial and boundary conditions, and ξ is the coordinate along which the acoustic pressure varies, i.e. the direction along which the acoustic disturbance travels. This solution is the sum of two waves travelling at speed c in the $+\xi$ and $-\xi$ direction respectively.
In an arbitrarily oriented coordinate frame, if $\mathbf{n}$ is the unit vector in the direction of increasing ξ, then at a point $\mathbf{x}$ we can write $\xi = \mathbf{n} \cdot \mathbf{x}$. If one assumes, as is usually appropriate from physical considerations, that there

exists a time t_0 in the past before which the wave hasn't arrived and all field quantities are zero (***causality***), then the solution reduces to waves travelling in the positive direction:

$$p = f\left(t - \frac{\mathbf{n}\cdot\mathbf{x}}{c}\right) \ . \tag{1.16}$$

For an acoustic disturbance of constant frequency, the field variables oscillate sinusoidally with time, so

$$p = |A|\, cos(\omega t - \varphi) = \mathrm{Re}\{Ae^{(i\varphi - i\omega t)}\} \ , \tag{1.17}$$

$$\begin{aligned} \text{where} \quad \omega \ &= \text{angular frequency,} \\ \varphi \ &= \text{phase,} \\ \text{and we have} \quad T \ &= \frac{2\pi}{\omega} = \text{period,} \\ f \ &= \frac{\omega}{2\pi} = \text{frequency.} \end{aligned}$$

If a sinusoidal wave $p = |A|\, cos(\omega t)$ travels in the $\mathbf{n}$ direction, then we must have $p = f(t - \mathbf{n}\cdot\mathbf{x}/c)$, and consequently

$$p = |A| \cos\left[\omega(t - \frac{\mathbf{n}\cdot\mathbf{x}}{c})\right] = \mathrm{Re}\{e^{-i\omega(t - \frac{\mathbf{n}\cdot\mathbf{x}}{c})}\} = \mathrm{Re}\{e^{i\mathbf{k}\cdot\mathbf{x} - \omega t}\} \ , \tag{1.18}$$

where we have used the *wavevector* $\mathbf{k} = \frac{\omega}{c}\mathbf{n}$. The above rightmost expression is the one usually and most conveniently used in practical calculations.

NOTE: Even though the physical quantity is given by the real part only, full complex waves are normally used in calculations, and the real part is subsequently taken as appropriate. Consequently, if the acoustic field is expressed in terms of a complex velocity potential ψ, we should be careful to take

$$\begin{aligned} p &= \mathrm{Re}\left[i\omega\rho\psi\exp(-i\omega t)\right] \\ \mathbf{v} &= \mathrm{Re}\left[\nabla\psi\exp(-i\omega t)\right] \end{aligned} \tag{1.19}$$

when dealing with real physical quantities.
Any acoustic disturbance $p(\mathbf{x}, t)$ can be written as a superposition of time-harmonic waves 1.17. This can be done using a Fourier transform (as long as $\mid p(\mathbf{x}, t) \mid$ and $\mid p(\mathbf{x}, t) \mid \in L^2$):

$$p(\mathbf{x}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} p(\mathbf{x}, \omega)\exp(-i\omega t)d\omega \tag{1.20}$$

where
$$p(\mathbf{x}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\mathbf{x}, t) \exp(i\omega t) dt \tag{1.21}$$

If we substitute a harmonic wave $p = e^{i\mathbf{k}\cdot\mathbf{x}-\omega t}$ in the wave equation (1.11) (noting that $\mathrm{Re}\{\cdot\}$ and $\frac{\partial}{\partial t}\{\cdot\}$ commute), we obtain

$$\frac{\omega^2}{c^2} p + \nabla^2 p = 0 \ ,$$

or, by using the *wavenumber* $k = \frac{\omega}{c}$

$$\nabla^2 p + k^2 p = 0 \ . \tag{1.22}$$

This form of the wave equation, suitable for time-harmonic waves, is usually called the **Helmholtz equation**, or **reduced wave equation**.

When considering time-harmonic problems then, it is usual (and obviously very convenient) to drop the time-dependent part of the wave altogether. This is possible, at least for part of the calculations, in the case of a non-monochromatic wave, by decomposing it into monochromatic waves using Fourier analysis. Since the wave equation is linear, each Fourier component obeys the Helmholtz equation, and the total field can be reconstracted after solving the scattering problem for whatever boundary conditions on any finite surfaces are appropriate. In this case, though, it is not possible to express the causality condition in the same way as before. Causality then is expressed by the integrability condition implicit in assuming that a Fourier representation of the wave exists. What was introduced as a condition in time (initial value), and cannot in that form be readily applied to a superposition of stationary waves, is equivalent to a condition in space (boundary condition at infinity):

$$p(\mathbf{x}) = \mathcal{O}(|\ \mathbf{x}\ |^{-1/2}) \tag{1.23}$$

or, more usually:
$$|\mathbf{x}| \left(\frac{\partial p(\mathbf{x})}{\partial |\mathbf{x}|} - ikp(\mathbf{x}) \right) \to 0 \tag{1.24}$$

uniformly as $|\ \mathbf{x}\ | \to \infty$. This is the ***Sommerfeld radiation condition***, and it expresses the requirement that the field should contain no incoming waves as $|\mathbf{x}| \to \infty$. In general, integrability, hence causality, will also result in restrictions imposed on the contour chosen for the integration in the complex plane.

1.2 Electromagnetic waves

In this section the wave equation obeyed by *electromagnetic waves* is derived, and we introduce the general scattering problem for electromagnetic waves. We shall begin with Maxwell's equations for an electromagnetic field in a generic medium with permittivity ϵ and permeability μ, in SI units (also sometimes called MKS):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.25}$$
$$\nabla \cdot \mathbf{B} = 0 \tag{1.26}$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \tag{1.27}$$
$$\nabla \cdot \mathbf{D} = \rho\ , \tag{1.28}$$

Here $\mathbf{E}$ is the electric field intensity, $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the magnetic field intensity, $\mathbf{D}$ is the so-called electric displacement, $\mathbf{J}$ is the current density, and ρ is the electric charge density. These quantities are related by

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P} \tag{1.29}$$
$$\mathbf{B} = \mu \mathbf{H} + \mathbf{M}\ , \tag{1.30}$$

where $\mathbf{P}$ is the electric polarization and $\mathbf{M}$ the magnetization.
In free space, we have $\mathbf{P} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$, and Maxwell's equations reduce to

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \tag{1.31}$$
$$\nabla \cdot \mathbf{H} = 0 \tag{1.32}$$
$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.33}$$
$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}\ , \tag{1.34}$$

where ϵ_0 and μ_0 are the permittivity and permeability of free space respectively.
It is straightforward to see from the Maxwell equations that there exist ***scalar and vector potentials*** for the electromagnetic field. Since $\nabla \cdot \mathbf{B} = 0$, $\exists$ a vector field $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}\ . \tag{1.35}$$

Using this in the first of Maxwell's equations shows that $\mathbf{E}$ must satisfy:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}\ , \tag{1.36}$$

where V is a scalar field. $\mathbf{A}$ and V are not unique. It is always possible to find an arbitrary scalar Φ such that the vector

$$\mathbf{A_0} = \mathbf{A} - \nabla\Phi$$

also satisfies (1.35) giving the same $\mathbf{B}$, and the scalar

$$V_0 = V + \frac{\partial\Phi}{\partial t}$$

gives the same $\mathbf{E}$. This is a *gauge transformation*, and any particular choice of Φ is a *choice of gauge*.

We shall see that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ obey a wave equation equivalent to that derived in section 1.1 for acoustic waves.

Let us derive the wave equation first in free space, i.e. and in the case when there are no charges nor currents: $\rho = 0, \mathbf{J} = \mathbf{0}$. We shall start with equation (1.27), which in this case becomes:

$$\nabla\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} \tag{1.37}$$

Noting that $\nabla\times\{\cdot\}$ and $\frac{\partial}{\partial t}\{\cdot\}$ commute, if we now apply $\frac{\partial}{\partial t}$ to (1.37), and use equation (1.25), we obtain

$$\nabla\times(\nabla\times\mathbf{E}) = \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{1.38}$$

and, since $\nabla\cdot\mathbf{E} = 0$ in this case, and $\mu_0\epsilon_0 = c^{-2}$, where c is the speed of light, we arrive at the wave equation for $\mathbf{E}$

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0\ . \tag{1.39}$$

It is straightforward to derive a wave equation of the same form for the magnetic field $\mathbf{B}$. A wave equation for $\mathbf{E}$ can be similarly derived in the more general case were charges and currents are present, and has the form:

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = \epsilon_0^{-1}\nabla\rho + \mu_0\frac{\partial\mathbf{J}}{\partial t}\ , \tag{1.40}$$

where the r.h.s. represents source terms due to charges and currents. A similar equations for $\mathbf{B}$ also applies.

We notice here that the vector product $\mathbf{E}\times\mathbf{H}$ has the dimensions of an energy flux. It is indeed taken as the energy flow at a point (even though it is not unique), and is called ***Poynting vector***:

$$S = \mathbf{E}\times\mathbf{H} = \frac{1}{\mu}\mathbf{E}\times\mathbf{B} \tag{1.41}$$

The Poynting vector gives the direction of the energy flow. Similarly to acoustic plane waves, an electromagnetic plane wave shall be written $\mathbf{E}(\mathbf{r}, \mathbf{t}) = \mathbf{E}_0(t)e^{i\mathbf{k}\cdot\mathbf{r}}$, from which we can see that for plane waves the energy flow is perpendicular to the wavefront, and the energy travels in the direction of the wavevector $\mathbf{k}$. Note that, even though the functional form of an electromagnetic plane wave is the same as that of an acoustic plane wave, electromagnetic waves are *vector* waves, so all the equations are vector equations.
For a time-harmonic field $\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\}$ we can derive, as in the case of acoustic waves, a reduced wave equation: the Helmholtz equation for electromagnetic waves

$$\nabla^2\mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = 0 \ , \tag{1.42}$$

where $k^2 = \omega^2\mu\epsilon$.
The **radiation condition** for electromagnetic waves can be expressed (as before) in terms of the scalar and vector potentials, but is usually more conveniently expressed in terms of the field components:

$$\mid r\mathbf{E} \mid < K \ , \ \mid r\mathbf{H} \mid < K$$

$$r(\mathbf{E} + Z_0\hat{\mathbf{i}}_\mathbf{r} \times \mathbf{H}) \ \rightarrow \ 0 \ , \ \text{as} \mid \mathbf{r} \mid \rightarrow \infty \ , \tag{1.43}$$

$$r(\mathbf{H} - \hat{\mathbf{i}}_\mathbf{r} \times \mathbf{E}/Z_0) \ \rightarrow \ 0 \ , \ \text{as} \mid \mathbf{r} \mid \rightarrow \infty \ , \tag{1.44}$$

where $Z_0 = \sqrt{\mu/\epsilon}$ = ***impedance*** of the medium.

Polarized waves

Plane waves solutions of (1.40) or (1.39) and their equivalents for the magnetic field are again fundamental in practical applications, as in the case of acoustic waves, either because only far-field solutions are of interest, or because any wave can be represented as a superposition of plane waves.
Of particular interest are plane waves which are linearly polarized. Two kinds of linear polarizations are possible. Let's take Cartesian coordinates and a plane wave with direction of propagation $\mathbf{k}$ in the (x, y)-plane. Then, either the electric vector $\mathbf{E}$ is parallel to the z-coordinate:

$$\mathbf{E} = \hat{\mathbf{z}}E_z \quad , \quad \mathbf{E}\text{-polarization} \quad \text{or} \quad \text{TM wave} \tag{1.45}$$

or:

$$\mathbf{H} = \hat{\mathbf{z}}H_z \quad , \quad \mathbf{H}\text{-polarization} \quad \text{or} \quad \text{TE wave} \ . \tag{1.46}$$

When talking of "*direction of polarization*", one normally refers to the direction of **E** (but note that the opposite convention is sometime found in the literature). It is immediately apparent that in many scattering problems with linearly polarized waves, the vector wave equation will reduce to a scalar equation for either E_z or H_z.
For example, if a TM wave is incident on a surface that can be described by $S = f(\rho, \phi)$ in cylindrical polar coordinates, independently of z, then for this scattering problem the incident field is given by

$$\mathbf{E}^{inc} = \hat{\mathbf{z}} E_z^{inc} \ , \ \mathbf{H}^{inc} = -\frac{i}{kZ} \left(\frac{\partial E_z^{inc}}{\partial y} \hat{\mathbf{x}} - \frac{\partial E_z^{inc}}{\partial x} \hat{\mathbf{y}} \right) \ , \tag{1.47}$$

where $Z = \sqrt{\mu/\epsilon}$ is the surface impedence, and depends on the properties of the two media and the surface, and usually varies with the incoming field at each point. In general, Z is also a function of frequency and angle of incidence.
Since the boundary conditions are independent of z, then the scattered field must also be E-polarized, and of the form

$$\mathbf{E}^{sc} = \hat{\mathbf{z}} E_z^{sc} \ , \ \mathbf{H}^{sc} = -\frac{i}{kZ} \left(\frac{\partial E_z^{sc}}{\partial y} \hat{\mathbf{x}} - \frac{\partial E_z^{sc}}{\partial x} \hat{\mathbf{y}} \right) \ , \tag{1.48}$$

therefore the scattering problem reduces to finding the scalar function E_z^{sc}, and is analogous to the problem of an acoustic field scattered by a soft surface. Similarly, the case of H-polarization is analogous to that of an acoustic field scattered by a hard surfaces. All problems where the scatterer is axisymmetric and the incident electromagnetic field is polarized in the direction parallel to the axis of symmetry therefore reduce to a scalar problem.

1.3 Boundary conditions

The constraints imposed on the solutions of the wave equation at a surface must reflect the nature of the solid object defined by the surface or, if the surface in question is an interface between two fluids, the different characteristic properties of the two fluids.
If the surface is perfectly reflecting, (for acoustic waves) or perfectly conducting (for electromagnetic waves), i.e. the tangential component of the total electric field at the surface is zero:

$$\mathbf{E} - (\mathbf{E} \cdot \mathbf{n})\mathbf{n} = 0 \ , \tag{1.49}$$

then two cases are possible:
Neumann condition, when the normal derivative of the potential field is given at the boundary, i.e., if $\mathbf{n}$ is the unit normal pointing outward from the surface:

$$\frac{\partial \psi(\mathbf{r})}{\partial n} = 0, \ \mathbf{r} \text{ on } S. \tag{1.50}$$

For acoustic waves, this corresponds to an acoustically hard surface, or in the case of electromagnetic waves in 2D, to a vertically polarized electromagnetic wave on a perfectly conducting surface.
Dirichlet condition, when the value of the potential field is given at the boundary:

$$\psi(\mathbf{r}) = 0, \ \mathbf{r} \text{ on } S. \tag{1.51}$$

which, for acoustic waves, corresponds to a pressure-release or acoustically soft surface, and in the case of electromagnetic waves corresponds to a horizontally polarized electromagnetic wave in 2D on a perfectly conducting surface.
In most real cases the surfaces is neither perfectly reflecting, nor perfectly conducting, and the boundary condition is of mixed type:
Cauchy condition (also called ***Robin***, or ***impedance boundary condition***). In this case both the potential and its normal derivative are different from zero at the boundary, and the boundary condition is then expressed as an equation relating these two quantities:

$$\frac{\partial \psi}{\partial n}(\mathbf{r}) = iZ(\mathbf{r}, \omega, \theta, ...)\psi(\mathbf{r}) \ \mathbf{r} \text{ on } S. \tag{1.52}$$

For electromagnetic waves the boundary condition relates the tangential component of the electric field at the surface to the normal component of the magnetic field at the surface:

$$\mathbf{E} - (\mathbf{E} \cdot \mathbf{n})\mathbf{n} = Z(\mathbf{r}, \omega, \theta, ...)\mathbf{n} \times \mathbf{H} \ , \tag{1.53}$$

Here Z depends on the properties of the two media and usually varies with the incoming field at each point. In general, Z is also a function of frequency and angle of incidence.
The impedance boundary condition can also be expressed as

$$\mathbf{n} \times \nabla \times \mathbf{E}_T = iZ\mathbf{n} \times (\mathbf{E} \times \mathbf{n})$$

in a form similar to the one for scalar waves.
Exact boundary conditions at an interface between two media are given by the jump (continuity) conditions:

$$\begin{aligned} \rho_1 \psi_1 &= \rho_2 \psi_2 \\ \frac{\partial \psi_1}{\partial n} &= \frac{\partial \psi_2}{\partial n} \frac{\partial \psi}{\partial n}^{(2)} \end{aligned} \tag{1.54}$$

where the subscripts 1 and 2 refer to the two media, and we take $\mathbf{n}$ as the normal directed into medium 1.
For electromagnetic waves, the boundary conditions at an interface are continuity of the normal component of $\mathbf{B}$ and the tangential component of $\mathbf{E}$:

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0 \tag{1.55}$$

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0\ , \tag{1.56}$$

plus

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \rho_s \tag{1.57}$$

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}_S\ , \tag{1.58}$$

where ρ_s is surface charge and $\mathbf{J}_S$ surface current.

1.4 Green's functions

In most problems of practical interest in acoustics, there will be one or more sources of sound, and the space where the problem needs to be solved will include one or more surfaces. Consequently, the differential equation to be solved will be an *inhomogeneous* version of (1.11) or (1.22), and the solutions will be subject to other *boundary conditions* in addition to (1.24). In general the problem in question will then be defined by a differential equation

$$\nabla^2 p(\mathbf{x}, t) + k^2 p(\mathbf{x}, t) = f(\mathbf{x}, t) \ , \tag{1.59}$$

together with boundary conditions on one or more surfaces and the Sommerfeld conditions. It is usually not easy to find solutions for such boundary value problems, but the task is greatly facilitated by the use of an auxiliary function associated with the differential equation, known as ***Green's function***.

In order to illustrate the concept of a Green's function, and provide the means of constructing Green's functions for different problems, let's first write (1.59) in operator form as

$$Lp(\xi) = f(\xi) \ , \tag{1.60}$$

where L is a linear operator, p the unknown function, and f is a known function determined by the source. The variable ξ denotes a point in an n-dimensional space which can include time as one of the coordinates. The solution of (1.60) can be sought in principle by finding the inverse of the operator L,

$$p(\xi) = L^{-1} f(\xi) \ , \tag{1.61}$$

but this is so far not particularly useful in practice. Since L is a differential operator, if L^{-1} exists, it can be reasonably assumed to be an integral operator. If we assume that L^{-1} is an integral operator with kernel K, i.e. such that

$$L^{-1} f(\xi) = \int K(\xi, \eta) f(\eta) d\eta$$

for any functions f defined in the same domain as p, then we can write

$$p(\xi) = LL^{-1} p(\xi) = L \int K(\xi, \eta) p(\eta) d\eta \ ,$$

Since L is a differential operator with respect to the variable ξ, we can formally write

$$p(\xi) = \int LK(\xi, \eta) p(\eta) d\eta \ .$$

This can be true only if

$$LK(\xi,\eta) = \delta(\eta - \xi) \ , \tag{1.62}$$

in which case we can write the solution to (1.60) as

$$p(\xi) = \int K(\xi,\eta) f(\eta) d\eta \tag{1.63}$$

The kernel K of the operator L^{-1} is called the *Green's function* for the problem and will therefter be denoted by $G(\xi,\eta)$. We can see from (1.63) that its knowledge allows us to find the solution of the wave equation for any known source $f(\xi)$, at least in principle. Equation (1.62) shows that the Green's function is the field generated by a delta-function inhomogeneity, i.e. the solution of the inhomogeneous wave equation (1.60) with the source term $f = \delta(\eta - \xi)$.
Due to the symmetric property of G:

$$G(\xi,\eta) = G^*(\eta,\xi)$$

This reciprocity relation means that $G(\mathbf{x},\mathbf{y},t,t')$ can equivalently represent the field at a point $\mathbf{x}$ due to a 'disturbance' at $\mathbf{y}$, or the field at $\mathbf{y}$ due to a 'disturbance' at $\mathbf{x}$. In other words, the Green's function is unchanged if source and receiver are interchanged. We note that, with regard to the time coordinate, the reciprocity implies time reversal: $G(\mathbf{x},\mathbf{y},t,0) = G(\mathbf{y},\mathbf{x},0,-t)$, so causality is satisfied.
The Green's function defined above is not unique: it is always possible to add to it a solution of the homogeneous wave equation, and the result will of course still satisfy (1.62). The particular solution for the Green's function which is independent of any boundary conditions is called the *free space Green's function*, and shall usually be denoted by $G_0(\xi,\eta)$. Any other Green's function can be written as

$$G(\xi,\eta) = G_0(\xi,\eta) + G_H(\xi,\eta) \ , \tag{1.64}$$

where $G_H(\xi,\eta)$ is a solution of

$$L(\xi)G(\xi,\eta) = 0 \ . \tag{1.65}$$

When $G_H(\xi,\eta)$ is chosen to satisfy the boundary conditions for the problem, then $G(\xi,\eta)$ is the exact Green's function for the problem.

We shall derive here the free space Green's function for time-dependent wave equation in 1D, i.e. the function G satisfying:

$$\frac{\partial^2 Gx,t}{\partial t^2} - c^2 \frac{\partial^2 G(x,t)}{\partial x^2} = \delta(x-y)\delta(t-\tau) \tag{1.66}$$

If we Fourier transform (1.66) in both space and time, it becomes

$$-\omega^2\hat{G}(k,\omega) + c^2k^2\hat{G}(k,\omega) = e^{iky}e^{-i\omega\tau} \ , \tag{1.67}$$

so the transform of the required Green's function is given by

$$\hat{G}(k,\omega) = \frac{1}{c^2}\frac{e^{iky}e^{-i\omega\tau}}{k^2-\omega^2/c^2} \ , \tag{1.68}$$

and $G(x,t)$ can be obtained by transfoming back:

$$G(x,t) = \frac{1}{4\pi^2c^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-ik(x-y)}e^{i\omega(t-\tau)}}{k^2-\omega^2/c^2}dkd\omega \tag{1.69}$$

The integral in (1.69) must be calculated taking care that the contour of integration is chosen in a way that satisfies the causality condition. As discussed in section 1.1, this means requiring that the time-Fourier transformed function $G(x,\omega)$ must be analytic in $\mathrm{Im}(\omega) \leq 0$. Therefore, when integrating in the complex k-plane, we need to take the limit from below at the pole $k = \omega/c$, and the liimit from above at the pole $k = -\omega/c$. In the first case the contour will have a small indentation above the pole, in the second case, a small indentation below. With these contraints then, if we first carry out the inverse in k-space we obtain:

$$G(x,\omega) = \frac{1}{4\pi^2c^2}\int_{-\infty}^{\infty}\frac{e^{-ik(x-y)}}{k^2-\omega^2/c^2}dk = \frac{e^{-i\omega\frac{|x-y|}{c}}}{4\pi i\omega c} \ . \tag{1.70}$$

The inverse transform in time then gives:

$$G(x,t) = \frac{1}{4\pi ic}\int_{-\infty}^{\infty}\frac{e^{i\omega(t-\tau-\frac{|x-y|}{c})}}{\omega}d\omega = \frac{1}{2c}H\left(t-\tau-\frac{|\ x-y\ |}{c}\right) \ . \tag{1.71}$$

The time $(t-\tau-\frac{|x-y|}{c})$ is called ***retarded time***, and is the time at which the disturbance observed at (x,t) has been emitted by the source at (y).
In 3 dimensions, the free space Green's function for the time-dependent wave equation is

$$G(x,t) = \frac{1}{4\pi c^2r}\delta(t-\tau-r/c) \ , \tag{1.72}$$

where $r =|\ \mathbf{x}-\mathbf{y}\ |$,
and the free space Green's function for the Helmholtz equation is

$$G(x,t) = \frac{e^{ikr}}{4\pi r} \ . \tag{1.73}$$

The above represents a spherically symmetric wave, and can be derived as the wave generated by a source consisting of an oscillating sphere, in the limiting case where the radius tends to zero. Such source is called a ***point source***, or ***monopole***. In the case of electromagnetic waves, a point source is equivalent to a charge.

It is instructive to consider a source $Q(\mathbf{r})$, uniformly distributed within a sphere. The Helmholtz equation for the wave field is then

$$\nabla^2 p(\mathbf{x},\omega) + k^2 p(\mathbf{x},\omega) = Q(\mathbf{x}) \ . \tag{1.74}$$

This can now be written, using (1.63), as:

$$p(\mathbf{x},\omega) = \frac{1}{4\pi}\int \frac{e^{ikr}}{r}\ Q(\mathbf{y})dy \tag{1.75}$$

If the radius of the sphere $\mathbf{r}'$ is very small, so $r' \ll r$, then we can expand $(e^{ik|r-r'|}/(\mid r-r'\mid)$ in a power series:

$$\begin{aligned}(e^{ik|r-r'|)}/(\mid r-r'\mid) \ &= \\ \frac{e^{ikr}}{r} - \mathbf{r}'\cdot\nabla\left(\frac{e^{ikr}}{r}\right) \ &+ \ \frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\left(\frac{e^{ikr}}{r}\right) + \ldots\end{aligned}$$

If we substitute this expansion in (1.75), we obtain:

$$p = Q_0\frac{e^{ikr}}{r} + Q_i\frac{e^{ikr}}{r^2} + Q_{ij}\frac{e^{ikr}}{r^3} + \ldots \tag{1.76}$$

The coefficients Q_0, Q_i and Q_{ij} (obtained by integrating over the volume of the sphere containing the sources), are called respectively ***monopole***, ***dipole*** and ***quadrupole*** strength, and the series just obtained **multipole expansion**.

1.5 The Kirchoff-Helmholtz equation

By using the Green's function it is possible to derive an integral form of the Helmholtz equation, which facilitates calculations of sound propagation and scattering, and allows sources and boundary conditions to be treated in a simple and convenient way.
In order to derive this integral equation, we shall first recall the following vector identities. Given any two function f and g, we have:

$$\nabla \cdot (f\nabla g) = f\nabla^2 g + (\nabla f)\cdot(\nabla g) \ . \tag{V1}$$

If $f\nabla g$ is a vector field continuously differentiable to first order, which we shall denote by $\mathbf{F} = f\nabla g$, then we can apply to it the following theorem, which transforms a volume integral into a surface integral:
Gauss theorem *If V is a subset of $\mathbb{R}^n$, compact and with piecewise smooth boundary S, and $\mathbf{F}$ is a continuously differentiable vector field defined on v, then*

$$\int_V \nabla\cdot\mathbf{F}\, dV = \int_S \mathbf{F}\cdot\mathbf{n}\, dS \ , \tag{V2}$$

where $\mathbf{n}$ is the outward-pointing unit normal to the boundary S.
In $\mathbb{R}^3$, for an $\mathbf{F}_1 = f\nabla g$ and an $\mathbf{F}_2 = g\nabla f$, we have, using V2 and V1:

$$\int_V \left[f\nabla^2 g + (\nabla f)\cdot(\nabla g)\right]\, dV = \int_{\partial V} f\nabla g\cdot\mathbf{n}\, dS \ , \tag{1.77}$$

$$\int_V \left[g\nabla^2 f + (\nabla g)\cdot(\nabla f)\right]\, dV = \int_{\partial V} g\nabla f\cdot\mathbf{n}\, dS \ , \tag{1.78}$$

and subtracting (1.78) from (1.77) we obtain:

$$\int_V \left(f\nabla^2 g - g\nabla^2 f\right)\, dV = \int_{\partial V} (f\nabla g - g\nabla f)\cdot\mathbf{n}\, dS \ . \tag{1.79}$$

This result can be used can be used to solve a general scattering problem, involving one or more sources and write the solution in terms of the (unknown) field and its normal derivative along the boundary. The integral equations obtained can in principle be solved to find these unknown surface field values. This approach applies whether the problem involves an interface with a vacuum or with a second medium.
Consider first a finite region V contained between two smooth closed surfaces S_0 and S_1, and containing a source $Q(\mathbf{r})$.

.

Let G be the free space Green's function, and ψ the solutions to the inhomogeneous equation

$$\nabla^2\psi + k^2\psi = Q(\mathbf{r}) \; . \tag{1.80}$$

Using the vector identities introduced above, we can write:

$$\int_V \left(\psi\nabla^2 G - \nabla^2\psi G\right) dV = \int_{S_0+S_1} \left(\psi\frac{\partial G}{\partial n} - \frac{\partial\psi}{\partial n}G\right) ds \; , \tag{1.81}$$

where we have used $d/d\mathbf{n} = \mathbf{n}\cdot\nabla$. If we let the outer surface S_1 go to infinity, then, provided ψ obeys the Sommerfeld boundary condition at infinity, then the integral over S_1 vanishes.
Substituting in (1.81) the expressions for $\nabla^2\psi$ and $\nabla^2 G$ obtained by the appropriate wave equations, i.e.

$$\begin{aligned}\nabla^2 G &= \delta(\mathbf{r}-\mathbf{r}') - k^2 G \\ \nabla^2\psi &= Q(\mathbf{r}) - k^2\psi\end{aligned}$$

we obtain

$$\int_V \psi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') - \; Q(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}' = \int_{S_0+S_1} \left(\psi\frac{\partial G}{\partial n} - \frac{\partial\psi}{\partial n}G\right) ds \; . \tag{1.82}$$

But

$$\psi_i(\mathbf{r}) = \int_V Q(\mathbf{r}')G(\mathbf{r},\mathbf{r}')d\mathbf{r}'. \tag{1.83}$$

is the incident field ψ_i inside the volume V. Using this result, then, we can write (1.81) as

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_{S_0} \left[\psi(\mathbf{r}_0)\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - \frac{\partial\psi}{\partial n}(\mathbf{r}_0)G(\mathbf{r},\mathbf{r}_0)\right] d\mathbf{r}_0 \; . \tag{1.84}$$

This is the **Kirchoff-Helmholtz equation**, an integral (implicit) form of the Helmholtz equation, which is of great practical use in calculating the field induced by sources scattered by finite boundaries.

1.5 The Kirchoff-Helmholtz equation

2 Canonical cases

2.1 Scattering from a flat surface

Consider first a time-harmonic scalar (acoustic) wave potential $\psi \exp(-i\omega t)$ of frequency ω in a 2-dimensional medium (x, z), having density ρ, sound-speed c. In what follows the time variation $e^{-i\omega t}$ will be suppressed. It is useful to start by considering the elementary problem of plane wave reflection by flat boundaries. Suppose therefore that the boundary between two media (say medium 1 and medium 2) is an infinite flat surface at $z = 0$. and suppose ψ is the solution due to a wave ψ_i incident on this in the first medium:

$$\psi_i = \exp(ik[x \sin\theta - z\cos\theta])$$

with reflected and transmitted fields ψ_s, ψ_T respectively.
Define the *scattered field* $\psi_s(x, z)$ in medium 1 by

$$\psi_s = \psi - \psi_i \ . \tag{2.1}$$

Then ψ, ψ_i, ψ_s obey the wave equation

$$\left(\nabla^2 + k^2\right)\psi = 0 \ . \tag{2.2}$$

Suppose the lower medium has density ρ_2, wavespeed c_2 and corresponding wavenumber $k_2 = \omega/c_2$. Denote the total *transmitted field* in medium 2 by $\psi^{(2)}$, so that this obeys the wave equation

$$\left(\nabla^2 + k^2\right)\psi^{(2)} = 0 \ . \tag{2.3}$$

By the wave equation above and the radiation condition (i.e. reflected and transmitted waves consist of outgoing waves only), the scattered wave has the form of specular reflection

$$\psi_s = R(\theta)e^{i(kx\sin\theta + kz\cos\theta)} \tag{2.4}$$

where R is the reflection coefficient. If the lower medium is not a vacuum we also have

$$\psi_T = T(\theta)e^{i(kx\sin\theta - zq)} \tag{2.5}$$

where

$$q = \sqrt{k_2^2 - k^2\sin^2\theta}.$$

Writing $\psi = \psi_i + \psi_s$ and using jump conditions (1.54) we obtain

$$\begin{aligned} R &= \frac{\alpha - 1}{\alpha + 1} \\ T &= \frac{2\rho\alpha}{\rho_2(\alpha+1)} = \frac{2k\cos\theta}{q(\alpha+1)} \end{aligned} \tag{2.6}$$

where

$$\alpha = \frac{\rho_2 k\cos\theta}{\rho q}$$

The two perfectly reflecting cases can be recovered from this as $\rho_2 \to 0$ or $\rho_2 \to \infty$, or by using the boundary conditions (1.51), (1.50) directly in (2.4). Thus Dirichlet ($\psi = 0$) gives $R = -1$, and Neumann becomes $R = +1$.

Total internal reflection: Suppose that the sound-speed is greater in medium 2, $c_2 > c$. Then $k_2 < k$ and we can have total internal reflection. From (2.5)

$$q^2 = k^2\left[\left(\frac{c}{c_2}\right)^2 - \sin^2\theta\right]$$

so that when $\theta > \arcsin(c/c_2)$ (critical angle) q becomes imaginary, and we shall write $q = ia(\theta)$ where $a(\theta)$ is real. We must choose the positive root to obey radiation conditions, and ψ_T is an ***evanescent*** wave, i.e. it decays exponentially away from the boundary,

$$\psi_T = Te^{ikx\sin\theta + za}.$$

(This form is also referred to as an ***inhomogeneous*** plane wave, and it may be considered as propagating at a complex rather than a real angle.) We find in particular that at the critical angle q vanishes, and we have

$$R = 1,\ T = 2.$$

Thus the interface appears as rigid when viewed from medium 1, but excites a non-zero transmitted component. The question then arises of what happens to the energy at the interface.

Energy flux

Since we will be looking for solutions to the scattering problem in terms of plane waves, it is important to consider the energy carried by plane waves across a given plane.

Remember that we started this course by deriving the wave equation from the linearised versions of the equations of mass and of momentum conservation for a fluid. An equivalent conservation equation for the energy can also be written, and is very helpful when trying to describe and understand the properties of sound fields.

Let us take the linearised momentum conservation equation in the absence of external forces (1.8), and dot multiply it by $\mathbf{v}$:

$$\rho_0 \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla p = 0 \ , \qquad (2.7)$$

where we have dropped the primes. Using now the vector identity A.4 and the linearised mass conservation equation in the absence of external sources (1.7), together with the relationship $p = c^2 \rho$, we can re-write this as:

$$\frac{\partial}{\partial t}\left[\left(\frac{1}{2}\rho v^2\right) - \left(\frac{1}{2}\frac{p^2}{\rho c^2}\right)\right] + \nabla \cdot (\mathbf{v}p) = 0 \qquad (2.8)$$

which can be interpreted as a conservation law for the energy, where

$$\begin{aligned} \frac{1}{2}\rho v^2 &= \text{acoustic kinetic energy density} \\ \frac{1}{2}\frac{p^2}{\rho c^2} &= \text{acoustic potential energy density} \\ \mathbf{v}p &= \text{acoustic energy flux} \end{aligned}$$

The relevant physical quantity of interest for time-harmonic fields is the time-averaged energy flux:

$$E = \frac{1}{T}\int_0^T \mathbf{v} p \, dt \ , \ \text{where } T = \frac{2\pi}{\omega} \qquad (2.9)$$

From 1.19 (which gives p and $\mathbf{v}$ in terms of the wave potential ψ) and since$\mathbf{n} \cdot \nabla = \frac{\partial}{\partial n}$, we can therefore calculate the time-averaged energy flux (at a point) in a direction n for a wave ψ:

$$E(\psi, n) = -\frac{\rho\omega}{2} \mathrm{Im}\left\{\psi^* \frac{\partial \psi}{\partial n}\right\} \qquad (2.10)$$

This is then integrated across the plane to which n is the normal to obtain the energy per unit area in the direction n. So for a homogeneous plane wave, say $\psi = \exp(ik[x\sin(\theta - z\cos(\theta)])$, the point-wise energy flux in the direction $\mathbf{n} = -z$, across some horizontal line in medium 2, is

$$E(\psi, n) = \frac{\rho_2 \omega k \cos(\theta)}{2} \tag{2.11}$$

Note that this is independent of x and z.
Now consider an inhomogeneous plane wave $\psi = \exp(ikx\sin(\theta) + az)$. Across the same horizontal boundary, we find that $\psi^* \partial\psi/\partial n$ is real, so that

$$E(\psi, n) \equiv 0,$$

and as we would expect this means that the transmitted field carries with it no energy.
(*Note*: What happens if another boundary is present somewhere below the first? In that case the radiation conditions are replaced by the appropriate boundary conditions for this interface. Then instead of just a single decaying wave, there may be in addition an exponentially growing part corresponding to the negative root of q above. The coefficient of this will depend on the boundary conditions and the depth of the layer, and the sum of these waves will again carry some non-zero energy, while the reflection coefficient R will no longer be unity.)
Finally consider the sum of two homogeneous plane waves, say

$$\psi = a_1 e^{ik(x\sin(\theta_1) - z\cos(\theta_1))} + a_2 e^{ik(x\sin(\theta_2) - z\cos(\theta_2))}.$$

We find that E is proportional to

$$|a_1|^2\cos(\theta_1) + |a_2|^2\cos(\theta_2) + 2\mathrm{Im}\left\{a_1 a_2^* e^{ik[(\sin(\theta_2)-\sin(\theta_1))x - (\cos(\theta_2)-\cos(\theta_1))z]}\right\}.$$

The last term is oscillatory and vanishes when spatially averaged, so that the energy in the plane waves *adds linearly*.

2.2 Scattering from a semi-infinite plane

When looking at scattering from a body with a finite boundary, additional restrictions may have to be applied. In particular, if the scatterer has a sharp edge, issues of uniqueness and singularity must be addressed. The restrictions on the field and its derivative near an edge are known as *edge conditions*. The appropriate requirement is that the energy in any finite region should be bounded, which is equivalent to requiring that the edge should not radiate energy of its own accord.

Following the derivation in [3], let's consider a 2-dimensional time-harmonic plane wave $\phi_{inc}\exp(-\omega t) = \exp(i(k_0 x\cos\theta + k_0 y\sin\theta - \omega t)$ incident on a semi-infinite surface defined by $y = 0$, $x > 0$. We shall remove the time factor $e^{-i\omega t}$ throughout. ϕ is an odd function of y, hence $\phi(x, 0) = 0$ for $x < 0$, and we may confine attention to the half space $y > 0$, since for $y < 0$, $\phi(x, y) = -\phi(x, -y)$. The governing equation is the Helmholtz equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^\phi}{\partial y^2} + k_0^2\phi = 0\ ,\ y \geq 0\ , \tag{2.12}$$

with boundary conditions on the plane $y = 0$

$$\phi = 0, \quad x < 0 \quad , y = 0\ , \tag{2.13}$$

$$\frac{\partial\phi}{\partial y} = -ik_0\sin\theta\exp(ik_0 x\cos\theta), \quad x > 0 \quad , y = 0\ , \tag{2.14}$$

and boundary conditions at infinity

$$\phi \sim -e^{ik_0 x\cos\theta + ik_0 y\sin\theta} + f_1(x)e^{ik_0 x}, \quad x \quad \to +\infty\ , \tag{2.15}$$

$$\phi \sim f_2(x)e^{-ik_0 x}, \quad x \quad \to -\infty\ , \tag{2.16}$$

for any fixed $y > 0$, where f_1 and f_2 are algebraic functions. The edge conditions are:

$$\phi \qquad \text{bounded} \tag{2.17}$$

$$|\ \phi\ | \ = O(x^\alpha)\ ,\text{for some } \alpha > -1,\ \text{ as } x \to \pm 0\ . \tag{2.18}$$

The solution is obtained by a straightforward application of the Wiener-Hopf technique. This is a method for solving partial differential equations by using the complex Fourier transform, and exploiting their analiticity properties. We shall therefore define full and half-range transforms as follows:

$$\Phi(s, y) = \int_{-\infty}^{+\infty}\phi(x, y)e^{isx}dx = \Phi_-(s, y) + \Phi_+(s, y)\ , \tag{2.19}$$

with

$$\Phi_+(s, y) \ = \ \int_0^{+\infty}\phi(x, y)e^{isx}dx \tag{2.20}$$

$$\Phi_-(s, y) \ = \ \int_{-\infty}^0\phi(x, y)e^{isx}dx\ . \tag{2.21}$$

If we assign a small imaginary part to $k_0 = k + i\varepsilon$, then we can see that $\Phi_+(s, y)$ is analytic for $s > -\varepsilon\cos\theta$, and $\Phi_-(s, y)$ is analytic for $s < \varepsilon$. The full-range Fourier transform (2.19) then is analytic in the strip:

$$S\ :\ -\varepsilon\cos\theta < s < \varepsilon\ . \tag{2.22}$$

It follows that the Fourier transform of (2.12) is

$$\frac{\partial^2 \Phi(s,y)}{\partial y^2} - s^2\Phi(s,y) + k_0^2\Phi(s,y) = 0 \ , \ s \text{ in the strip } S \ , \tag{2.23}$$

with solutions

$$\Phi(s,y) = e^{\pm\gamma y} \ , \tag{2.24}$$

where

$$\gamma = (s^2 - k_0^2)^1/2 \ , \ \gamma(0) = -ik_0 \tag{2.25}$$

has branch cuts from $\pm k_0$ to ∞ in the first and third quadrant. With this choice of cuts $\text{Re}\gamma \geq 0$ for all s in S.
The solution that remains bounded as $y \to \infty$ is

$$\Phi(s,y) = A(s)e^{-\gamma y} \ , \tag{2.26}$$

and $A(s)$ has to be determined from the boundary conditions 2.13 and 2.14, which give:

$$\Phi_-(s,0) = 0 \ , \tag{2.27}$$

$$\frac{\partial \Phi_+(s,0)}{\partial y} = \frac{k_0 \sin\theta}{s + k_0\cos\theta} \ . \tag{2.28}$$

Setting $y = 0$ in the solution (2.26) gives

$$\Phi_+(s,0) + \Phi_-(s,0) = A(s) \tag{2.29}$$

and the y-derivative at $y = 0$ gives

$$\left.\frac{\partial \Phi_+(s,y)}{\partial y}\right|_{y=0} + \left.\frac{\partial \Phi_-(s,y)}{\partial y}\right|_{y=0} = -\gamma(s)A(s) \tag{2.30}$$

Eliminating $A(s)$ from these equations, and using the boundary conditions, gives:

$$\gamma(s)\Phi_+ + \Phi'_- = \frac{k_0\sin\theta}{s + k_0\cos\theta} \ . \tag{2.31}$$

This equation can be written in the general (standard) form

$$K(s)U_+(s) + U_-(s) = P(s) \ , \tag{2.32}$$

known as **Wiener-Hopf equation**, where the *kernel* K is

$$K(s) = \gamma(s) = (s^2 - k_0^2)^{1/2} \ . \tag{2.33}$$

This is solved by the following steps:

Step 1: *Multiplication decomposition* of $K(s)$.

$$K(s) = K(s)_+ K_-(s)\ ,\ \alpha < \mathrm{Im}(s) < \beta\ , \tag{2.34}$$

where α and β are such that $K(s)_+$ is analytic in the half-plane $\mathrm{Im}(s) > \alpha$, and $K(s)_-$ is analytic in the half-plane $\mathrm{Im}(s) < \beta$, and

$$\begin{aligned} K(s)_+ &= \mathrm{O}(s^n) \text{ as } \mid s \mid \to \infty \text{in } \mathrm{Im}(s) > \alpha \\ K(s)_-^{-1} &= \mathrm{O}(s^m) \text{ as } \mid s \mid \to \infty \text{in } \mathrm{Im}(s) < \beta \end{aligned}$$

then the Wiener-hopf equation can be recast as:

$$K_+(s)U_+(s) + U_-(s)/K_-(s) = P(s)/K_-(s)\ , \tag{2.35}$$

Step 2: *Sum decomposition* of $R(s) = P(s)/K_-(s)$.

$$R(s) = R(s)_+ + R_-(s)\ , \tag{2.36}$$

where $R(s)_+$ and $R_-(s)$ are analytic and of algebraic growth in the respective half-planes.
Step 3: Completion
We can now write the Wiener-Hopf equation as

$$K_+(s)U_+(s) - R_+(s) = -U_-(s)/K_-(s) + R_-(s)\ . \tag{2.37}$$

Here the l.h.s. is analytic in the whole half-plane $\mathrm{Im}(s) > \alpha$, and the r.h.s. is analytic in the whole half-plane $\mathrm{Im}(s) < \beta$. So each side analytically continues the other to define a function $E(s)$ analytic in the whole s-plane. $E(s)$ can be found by using the conditions on the edge for the inverse transforms $u_+(x)$ and $u_-(s)$. The Abelian theorem relates the behaviour of $u_\pm(x)$ as $x \to \pm 0$ to the behaviour of $U_\pm(s)$ as $\mid s \mid \to \infty$.) If

$$\text{l.h.s.} = \mathrm{O}(s^n) \text{ and r.h.s.} = \mathrm{O}(s^m)$$

as $\mid s \mid \to \infty$ in their respective half-planes, then

$$E(s) = \text{ polynomial of degree N}\ ,$$

where N is the lesser of n, m, and the solution is complete.

In our case of scattering by semi-infinite plane, the factorization (Step 1) of K can be done by inspection, giving:

$$K_+K_- = (s + k_0)^{1/2}(s - k_0)^{1/2},\ \text{where} K_\pm = (s \pm k_0)^{1/2}. \tag{2.38}$$

The Wiener-Hopf equation (2.31) can be recast as (dividing by K_-)

$$(s+k_0)^{1/2}\Phi_+ + (s-k_0)^{-1/2}\Phi'_- = \frac{-k_0 \sin\theta}{(s+k_0\cos\theta)(s-k_0)^{1/2}} \; . \tag{2.39}$$

Now we need (Step 2) a sum decomposition of the r.h.s. of this equation, such that $R(s) = R(s)_+ + R_-(s)$. This can be done by

$$\frac{1}{(s+k_0\cos\theta)(s-k_0)^{1/2}} =$$

$$\frac{1}{(s+k_0\cos\theta)}\left[\frac{1}{(s-k_0)^{1/2}} - \frac{1}{(-k_0\cos\theta-k_0)^{1/2}}\right] +$$

$$\frac{1}{(-k_0\cos\theta-k_0)^{1/2}(s+k_0\cos\theta)} \; .$$

So the Wiener-Hopf equation can be written as:

$$(s+k_0)^{1/2}\Phi_+ \quad + \quad \frac{k_0\sin\theta}{(-k_0\cos\theta-k_0)^{1/2}(s+k_0\cos\theta)} =$$

$$-(s-k_0)^{1/2}\Phi'_- \quad -$$

$$\frac{k_0\sin\theta}{(s+k_0\cos\theta)} \qquad \left[\frac{1}{(s-k_0)^{1/2}} - \frac{1}{(-k_0\cos\theta-k_0)^{1/2}}\right] \equiv E(s)$$

This gives us the function $E(s)$, analytic in the whole complex plane, since the l.h.s. defines the analytic continuation in the upper plane, and the r.h.s. the analytic continuation in the lower plane.
We now need to find $E(s)$, or possibly Φ_+ and Φ'_-. Using the Abelian theorem), we can say something about the behaviour of Φ_+ and Φ'_- at ∞ from the behaviour of their inverse half-transform at 0. From the *edge conditions*, we have:

$\phi(x,0)$ bounded as $x \to +0$

$$\Rightarrow \Phi_+ = \mathrm{O}\left(\tfrac{1}{s}\right) \text{ as } |s| \to \infty, \text{ in } \mathrm{Im}(s) > -\varepsilon cos(\theta)$$

$$\left.\frac{\partial\phi_+(s,y)}{\partial y}\right|_{y=0} = \mathrm{O}(x^{\alpha}(\alpha > -1) \text{ as } x \to -0$$

$$\Rightarrow \Phi_- = \mathrm{O}\left(\tfrac{1}{s^{1+\alpha}}\right) \text{ as } |s| \;\to \infty, \text{ in } \mathrm{Im}(s) < \varepsilon$$

Therefore, $E(s) \to 0$ as $|s| \to \infty$, hence $E(s) \equiv 0$ by Liouville theorem. This tells us that our solution is unique, but the Wiener-Hopf equation still has 2 unknown: Φ_+ and Φ'_-. We can eliminate one of them by using the boundary conditions at the surface. Remember that the appropriate form of the solution $\Phi = \Phi_+ + \Phi_-$ is

$$\Phi = A(s)exp(-\gamma y) \ , \tag{2.40}$$

so at $y = 0$ we have $\Phi_+ + \Phi_- = A(s)$ (equation (2.26)). But the *boundary condition at the surface*:

$$\phi(x, 0) = 0 \ (x < 0) \tag{2.41}$$

gives $\Phi_-(s, 0) = 0$, so

$$\Phi_+(s) = A(s)$$

Using these results in the Wiener-Hopf equation we have:

$$\begin{aligned}\Phi_+ &= \frac{k_0 \sin\theta}{(s+k_0)^{1/2}(-k_0\cos\theta - k_0)^{1/2}(s+k_0\cos\theta)} = \\ &= \frac{-ik_0^{1/2}\sin\theta}{(s+k_0)^{1/2}(1+\cos\theta)^{1/2}(s+k_0\cos\theta)} = A(s)\end{aligned}$$

and we can inverse-transform the solution (2.40) to obtain

$$\phi(x, y) = \frac{-i}{2\pi}\frac{k_0^{1/2}\sin(\theta)}{(1+\cos\theta)}\int_C \frac{e^{-\gamma y - isx}}{(s+k_0)^{1/2}(s+k_0\cos\theta)} ds\ ' \tag{2.42}$$

where we have now taken the limit $\varepsilon \to 0$, and the integration path is along the real axis, avoiding the branch cuts from $-\mathbf{k}_0$ and $\mathbf{k}_0$, and the pole at $-\mathbf{k}_0 \cos(\theta)$.

Exact integration of (2.42) gives the total field due to scattering of an incident plane wave by the wedge. A contribution to this field will come from the pole at $-\mathbf{k}_0\cos\theta_0$. Writing $x = r\cos\theta$ and $y = r\cos\theta$, then letting $r \to \infty$ and deforming the integration path, it becomes clear that the pole contribution will be included only for some values of θ (i.e. only some observers), and it corresponds to the geometrical optics contribution. Only the result will be given here.

When $\theta > \theta_0$, the contribution from the pole is the plane wave

$$-e^{i(k_0 x\cos\theta_0 + k_0 y \sin\theta_0)} \ , \tag{2.43}$$

which cancels exactly with the incident wave;

When $\theta < -\theta_0$, it is the plane wave

$$-e^{i(k_0 x\cos\theta_0 - k_0 y \sin\theta_0)} \ , \tag{2.44}$$

i.e. the reflected wave. According to geometrical optics alone, there is a shadow zone (no field) for $\theta > \theta_0$, both reflected and incident waves for $\theta < -\theta_0$, and only the incident field when $-\theta_0 < \theta < \theta_0$, and therefore the field is discontinuous along the lines $\theta = \pm\theta_0$. Exact integration of (2.42) will give includes a *diffracted* field that penetrates the shadow zone, and is not discontinuous along $\theta = \pm\theta_0$.
It is possible to write the total field in terms of Fresnel integrals as follows:

$$\phi_T = \left(\frac{1}{\pi}\right)^{1/2} e^{\frac{i\pi}{4} - ikr} \quad \left[F((2kr)^{\frac{1}{2}} \sin\frac{1}{2}(\theta - \theta_0)) - F((2kr)^{\frac{1}{2}} \sin\frac{1}{2}(\theta + \theta_0))\right] ,$$

where

$$F(z) = \left(\frac{\pi}{2}\right)^{1/2} e^{iz^2} \left[e^{-\frac{i\pi}{4}} - (2\pi)^{\frac{1}{2}} \left(C\left(\frac{2^{\frac{1}{2}} z}{\pi^{\frac{1}{2}}}\right) - iS\left(\frac{2^{\frac{1}{2}} z}{\pi^{\frac{1}{2}}}\right)\right)\right]$$

and

$$C(z) = \int_0^z \cos\left(\frac{1}{2}\pi t^2\right) dt \; S(z) = \int_0^z \sin\left(\frac{1}{2}\pi t^2\right) dt$$

are the Fresnel integrals, related by:

$$C(z) - iS(z) = \int_0^z e^{\frac{1}{2}\pi t^2} dt \; .$$

When $|z| \gg 1$

$$F(z) = -\frac{i}{2z} + \frac{1}{4z^3} + \mathrm{O}\left(\frac{1}{|z|^5}\right) \; . \qquad (2.45)$$

When $|z| \ll 1$

$$F(z) = \frac{1}{2}\pi^{\frac{1}{2}} e^{-\frac{i\pi}{4}} - z + \mathrm{O}(|z|^2) \; . \qquad (2.46)$$

Near the edge, the total field behaves like

$$\phi_T \sim 1 - 2\left(\frac{2kr}{\pi}\right)^{\frac{1}{2}} e^{\frac{i\pi}{4}} \cos\frac{\theta_0}{2} \sin\frac{\theta}{2} \qquad (2.47)$$

Let's now look at the diffracted field (i.e. the field left after the geometrical optics contribution has been extracted). In the far field, it behaves like

$$\phi_{\text{diff}} \sim \left(\frac{2}{\pi kr}\right)^{\frac{1}{2}} e^{-\frac{i\pi}{4} - ikr} \frac{\sin\frac{\theta}{2} \cos\frac{\theta_0}{2}}{\cos\theta_0 - \cos\theta} , \qquad (2.48)$$

therefore it is a wave with cylindrical spreading $r^{-1/2}$ for $kr \gg 1$.
This asymptotic expression is not valid when θ is close to $\pm\theta_0$. In fact we can see from (2.42) that in this case the argument of F may be small even when $kr \gg 1$. The regions where this happens and the asymptotic expression above does not hold are regions along the lines $\pm\theta_0$, bounded by the parabolae $2kr\sin^2\frac{1}{2}(\theta \pm \theta_0) = \text{const}$. In these regions the full expression (2.42) must be used and there is no simpler approximation. The diffracted and the incident wave are of the same order of magnitude

2.3 Scattering from a wedge

The geometry of the wedge shall be defined as follows. The (infinite) wedge will be defined by the surfaces $y = 0$, $x > 0$ and $y = r \sin \beta$, $x = r \cos \beta$, where β is the exterior angle of the wedge and r the radial distance along the surface, and the edge of the wedge coincides with z-axis. In a cylindrical coordinate system $\{r, \theta, z\}$, one face of the wedge is at $\theta = 0$ and the other at $\theta = \beta$.

We shall consider scattering from the wedge due to a harmonic point source of strength Q at $r_s = (x_s, y_s, 0)$, and will seek a solution for the acoustic pressure field which satisfies the Helmholtz equation with Sommerfeld boundary conditions at infinity, and hard surface boundary conditions at the faces of the wedge:

$$\frac{\partial p}{\partial \theta} = 0 \text{ at } \theta = 0,\ \theta = \beta \ . \tag{2.49}$$

It is convenient to introduce a ***wedge index***

$$\nu = \frac{\pi}{\beta} \ , \tag{2.50}$$

and to express the distance between source and observer as a function

$$R(\varphi) = (r^2 + r_s^2 - 2 r r_s \cos \varphi + z^2)^{1/2} \ . \tag{2.51}$$

$R(\varphi)$ is the distance in the free-space Green's function

$$G(\varphi) = \frac{e^{ikR(\varphi)}}{R(\varphi)} \ . \tag{2.52}$$

The acoustic field at a point (r, θ, z) due to the source, without any surfaces present, is

$$p_0(r, \theta, z) = Q \frac{e^{ikR(\theta - \theta_s)}}{R(\theta - \theta_s)} \ . \tag{2.53}$$

We shall define $R(\varphi)$ so that R is positive for real φ, and R is analytic except at branch cuts, extending from branch points above and below the real axis.

We can see from (2.51) that the branch points are at

$$\varphi = 2\pi l \pm i\alpha \text{ where } \quad l \quad \text{is any integer}$$
$$\alpha \quad = \cosh^{-1} \frac{r^2 + r_s^2 + z^2}{2rr_s}$$

Since $G(\theta - \theta_s)$ satisfies the Helmholtz equation, from the principle of superposition we know that, given a position-independent contour C and function f,

$$p = \int_C f(\varphi) G(\varphi - \theta) d\varphi$$

or

$$p = \int_{C_\theta} f(\varphi + \theta) G(\varphi) d\varphi$$

(where we have made the change of variable $\delta = \varphi - \theta$, and renamed $\delta \equiv \varphi$) will also satisfy the Helmholtz equation.
We need to find an appropriate function $f(\varphi + \theta)$ and contour C.

Let's start with the simpler case of a

wedge with integer index
If ν = integer, then the problem can be solved with the *method of images*. The symmetry of this problem will require $2\nu - 1$ image sources, located periodically around the edge of the wedge at angles $2m\frac{\pi}{\nu} - \theta_s$ and $2m\frac{\pi}{\nu} + \theta_s$, $m = 0, 1, \dots (\nu - 1)$.
The solution is therefore:

$$p = Q \sum_{m=0}^{\nu-1} \left[G(2m\frac{\pi}{\nu} - \theta_s - \theta) + G(2m\frac{\pi}{\nu} + \theta_s - \theta) \right] \ . \qquad (2.54)$$

Alternatively we can express the sum above as a contour integral

$$p = \frac{Q}{2\pi i} \int_C G(\varphi) \left[h(\varphi + \theta + \theta_s) + h(\varphi + \theta - \theta_s) \right] d\varphi \ , \qquad (2.55)$$

where the function h has poles at $\varphi = 2m\frac{\pi}{\nu}$ with residue = 1 at each. The contour C should enclose one pole each for which $m = 0, 1, \dots (\nu - 1)$, for example all poles between $-\pi$ and π.
A suitable choice for $h(\varphi)$ is:

$$h(\varphi) = \frac{\nu}{2} \cot \left(\frac{\nu}{2} \right) \ . \qquad (2.56)$$

The residue at the poles $\varphi = 2m\frac{\pi}{\nu}$ is 1. The integral (2.55) form is useful for extending this result to non-integer ν.
The closed-contour choice for C is unsuitable for non-integer ν, though, because in that case the number of enclosed poles varies with θ. We will need to find a different contour that does not cross the real axis.
Note that, since the integrand is periodic of period 2π, integration along the downward infinite path from $\pi + i\infty$ to $\pi - i\infty$ will cancel exactly integration along the upward path from $-\pi - i\infty$ to $-\pi| + i\infty$. Therefore adding these paths to the integral (2.55) will leave it unchanged. We shall therefore:
1. add the two infinite downward and upward paths;
2. deform the upper arc of the closed-circuit path in such a way that it continues from the downward path from $\pi + \imath\infty$ and joins the upward path going to $-\pi| + \imath\infty$;
3. deform the lower arc of the closed-circuit path in such a way that it continues from the upward path from $-\pi - \imath\infty$ and joins the downward path going to $\pi| - \imath\infty$;
$\Rightarrow$ we have constructed an integration path split into two contours, C_U and C_L, neither of which crosses the real axis. C_U must pass below the branch points at $\varphi = i\alpha$, and C_L must pass above the branch points at $\varphi = -i\alpha$.

Extension to non-integer wedge indices
It is claimed that the solution (2.55), with the function $h(\varphi)$ in the integrand defined by (2.56), and where the contour of integration C is $C_U + C_L$, is also the solution of the scattering problem for a wedge with non-integer index ν.

This can be confirmed by verifying that this solution satisfies the Helmholtz equation and all the required boundary conditions. To do this, it is convenient to express (2.55) as

$$p = \frac{Q}{2\pi i}\int_{C_L} G(\varphi)\sum h d\varphi \ , \tag{2.57}$$

where

$$\sum h = \sum_{n,m=1}^{2} \frac{nu}{2} \cot\left(\frac{nu}{2}(\varphi + (-1)^n\theta + (-1)^m\theta_s)\right) \ . \tag{2.58}$$

We can do this, because $R(\varphi) = R(-\varphi)$, and the contour C_U is the inversion of C_L.

We can now verify:
(1) - The solution (2.57) satisfies the Helmholtz equation.
Since $G(\varphi)$ satisfies the Helmholtz equation:

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2} + k^2\right) G(\varphi) = 0 \ , \tag{2.59}$$

it follows that

$$(\nabla^2 + k^2)p = \frac{Q}{2\pi i}\int_{C_L} \frac{1}{r^2}\left[G(\varphi)\frac{\partial^2}{\partial\theta^2}\sum h - \sum h \frac{\partial^2 G}{\partial\varphi^2}\right] d\varphi \ , \tag{2.60}$$

and the r.h.s. vanishes.
(2) - The solution (2.57) obeys the boundary conditions at the edge faces:

$$\frac{\partial p}{\partial\theta} = 0 \text{ at } \theta = 0, \ \theta = \beta \ .$$

This follows from the expression (2.58), and because $h(\varphi)$ is an odd function of its argument, so $\sum h$ is even in θ for fixed φ, and the derivative is zero at $\theta = 0$. The derivative is zero also at $\theta = \beta$, because $h(\varphi)$ is periodic in φ with period 2β.
(3) - The solution (2.57) obeys the Sommerfeld radiation condition. This is because the Green's function will impose the correct limiting behavior to the integral for any function $\sum h$.
(4) - The solution (2.57) exhibits the right behaviour when approaching the singular point at the source location. When approaching the source, i.e. in the limit $r \to r_s, z \to 0, \theta \to \theta_s$, then $\alpha \to 0$, and a pole of $h(\varphi + \theta - \theta_s)$ approaches the origin. It is possible to isolate the contribution from this pole using an appropriate contour, and it gives $QG(\theta_s - \theta)$, which is just the direct wave from the source.

As before, the contribution from the poles gives the geometrical optics contribution to the field, and what is left over after extracting the solutions obtained from geometrical optics is the diffracted field.

2.3 Scattering from a wedge

General observations:
The contribution from the poles yields terms representing spherical waves diverging from an image, which correspond to a possible ray path connecting source and observer.
If $\beta < \pi$, field is accounted for by geometrical optics
If $\beta > \pi$, geometrical optics leads to field discontinuities at the boundaries of the shadow zone, and evaluation of all contributions to the integral is needed to estimate the acoustic field.
There is no diffracted wave contribution if the index ν is integer.
The case of the infinite half-plane is recovered in the limit $\beta \to 2\pi$.

3 Approximations

In general, the solution of most scattering problems can only be expressed analytically as some kind of integral, or as an implicit integral equation. Calculation of the actual values of the field then has to be obtained by computationally intensive numerical solutions. For many problems, though, it is possible to obtain approximate analytical solutions. We shall review the main ones in this chapter.

3.1 Born Approximation

The Born approximation is based on expressing the total wave field ψ, which is in general the solution of a scattering problem in a volume with sources and surfaces, as the sum of the incident field plus a 'small' perturbation:

$$\psi = \psi_i + \psi_s \ , \tag{3.1}$$

The actual solution in this approximation will take various forms, depending on how the perturbation is expressed.
We can immediately see how the Born approximation can be applied to the integral form of the wave equation (1.84), to obtain a first Born approximation

$$\psi^{(1)}(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_{S_0} \left[\psi_i(\mathbf{r}_0) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - \frac{\partial \psi_i}{\partial n}(\mathbf{r}_0) G(\mathbf{r}, \mathbf{r}_0) \right] d\mathbf{r}_0 \ , \tag{3.2}$$

and higher terms can be obtaind by iteration.
The Born approximation will only be valid when $\psi_s \ll \psi_i$, which intuitively must apply to some kind of 'weak scattering'. In order to understand better what this means in practice, to relate it to the physical features of a scattering problem, and find boundaries for its range of validity, we shall derive it here for some particular cases.
We shall consider the case where the scattered field is the result of a varying refractive index $n(\mathbf{r})$. The total field satisfies

$$\nabla^2 \psi + k^2(\mathbf{r}) \psi = 0 \ . \tag{3.3}$$

We can then write

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0(1 + n_\delta(\mathbf{r})) \ , \tag{3.4}$$

where it is assumed $n_\delta(\mathbf{r}) \ll 1$. Substituting $k_0 n(\mathbf{r})$ into (3.3) we get:

$$\nabla^2 \psi + k_0^2(\mathbf{r}) \psi = -k_0^2(n^2(\mathbf{r}) - 1)\psi \equiv -V(\mathbf{r})\psi \ . \tag{3.5}$$

Using (3.1), and the fact that the incident field satisfies

$$\nabla^2 \psi_i + k^2(\mathbf{r})\psi_i = 0 \ , \tag{3.6}$$

we can write the wave equation for the scattered wave

$$\nabla^2 \psi_s + k^2(\mathbf{r})\psi_s = -V(\mathbf{r})\psi \ . \tag{3.7}$$

We can then solve for ψ_s using the free space Green's function, with $-V(\mathbf{r})\psi$ as the source term

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ . \tag{3.8}$$

But $\psi_s = \psi - \psi_i$, so

$$\psi = \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ . \tag{3.9}$$

We can write the above implicit integral equation as an infinite series of explicit integral equations by forming successive approximations starting from the unperturbed incident field ψ_i:

$$\begin{aligned}
\psi^{(0)} &= \psi_i \\
\psi^{(1)} &= \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}')[V(\mathbf{r}')\psi^{(0)}(\mathbf{r}')]d\mathbf{r}' \\
\psi^{(2)} &= \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}')[V(\mathbf{r}')\psi^{(1)}(\mathbf{r}')]d\mathbf{r}' \\
\psi^{(3)} &= \ldots
\end{aligned}$$

The first iteration in this series, $\psi^{(1)}$, is know as the *first-order Born approximation*, usually referred to just as Born approximation.
This can also be put in a more compact form by writing the integration with Green's function as an operator:

$$\int G(\mathbf{r} - \mathbf{r}')[f(\mathbf{r}')]d\mathbf{r}' \equiv \hat{G}f$$

so (3.9) becomes $\psi = \psi_0 - \hat{G}V\psi$, and the series becomes

$$\begin{aligned}
\psi^{(0)} &= \psi_i \\
\psi^{(1)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} \\
\psi^{(2)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} + \hat{G}V\hat{G}V\psi^{(0)} \\
&\ldots \\
\psi^{(n)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} + \cdots + (\hat{G}V)^n\psi^{(0)}
\end{aligned}$$

This form of the Born series helps visualising the structure of the n-th order approximation, and is the one usually found in quantum mechanics, for scattering of a wave on a potential V.
Naturally the (first-order) Born approximation is good only if the first correction is smaller than the incident field, and in general will be valid only if the series converges.
Note: in the Born approximation, if the wave is expressed as a sum of incident and diffracted secondary wave, the scattering of the secondary wave is neglected. So no multiple scattering.

3.2 Rytov Approximation

The Rytov approximation is obtained by representing the total field as a complex phase:

$$\psi(\mathbf{r}) = e^{\phi(\mathbf{r})} \ . \tag{3.10}$$

Then, from the Helmholtz wave equation for ψ we have:

$$\nabla^2 e^{\phi(\mathbf{r})} + k^2 e^{\phi(\mathbf{r})} \tag{3.11}$$

Since

$$\nabla^2 e^{\phi(\mathbf{r})} = \nabla^2 \phi e^{\phi(\mathbf{r})} + (\nabla\phi)(\nabla\phi) e^{\phi(\mathbf{r})} \ ,$$

we get the following Riccati equation for the phase $\phi(\mathbf{r})$:

$$\nabla^2 \phi + (\nabla\phi)(\nabla\phi) + k^2 = 0 \ . \tag{3.12}$$

Let us now again write the refractive index as

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0(1 + n_\delta(\mathbf{r})) \ . \tag{3.13}$$

The field for $n(\mathbf{r}) = 1$, i.e. the field in a non-refractive medium, can be written as $\psi_i(\mathbf{r}) = e^{\phi_i(\mathbf{r})}$; it is of course the incident field, and its phase will satisfy

$$\nabla^2 \phi_i + (\nabla\phi_i)^2 + k_0^2 = 0 \tag{3.14}$$

If we write $\phi = \phi_i + \phi_s$ and subtract (3.14) from (3.12), we get

$$\nabla^2 \phi_s + 2(\nabla\phi_i)(\nabla\phi_s) = -\left((\nabla\phi_s)(\nabla\phi_s) + k_0^2(n^2 - 1)\right) \ . \tag{3.15}$$

Now, using the identity

$$\nabla^2(\psi_i \phi_s) = (\nabla^2 \psi_i)\phi_s + 2\psi_i(\nabla\phi_i)(\nabla\phi_s) + \psi_i \nabla^2 \phi_s \ ,$$

equation (3.15) becomes:

$$\nabla^2(\psi_i\phi_s) + k^2\psi_i\phi_s = \big((\nabla\phi_s)(\nabla\phi_s) + k_0^2(n^2-1)\big)\,\psi_i\ , \tag{3.16}$$

whose solution can be written as an integral using the free-space Green's function, to give:

$$\phi_s(\mathbf{r}) = \frac{1}{\psi_i(\mathbf{r})}\int G(\mathbf{r}-\mathbf{r}')\left[(\nabla\phi_s(\mathbf{r}'))(\nabla\phi_s(\mathbf{r}')) + k_0^2(n^2(\mathbf{r}')-1)\right]\psi_i(\mathbf{r}')d\mathbf{r}' \tag{3.17}$$

This equation is exact, but it's implicit and in practice provides no solution as it is. If we assume that the scattered phase ϕ_s is very small, then we can neglect $(\nabla\phi_s)^2$, and we obtain an approximate solution for the scattered phase

$$\phi_s(\mathbf{r}) \simeq \frac{1}{\psi_i(\mathbf{r})}\int G(\mathbf{r}-\mathbf{r}')[k_0^2(n^2(\mathbf{r}')-1)]\psi_i(\mathbf{r}')d\mathbf{r}' \tag{3.18}$$

The corresponding solution for the total field is then

$$\psi(\mathbf{r}) \simeq \psi_i(\mathbf{r})e^{\phi_s}\ . \tag{3.19}$$

This approximation is known as the (first) ***Rytov approximation***. It corresponds to taking the first order term in an infinite power series expansion of the phase $\phi(\mathbf{r})$. It is valid when $(\nabla\phi_s)^2 \ll k_0^2(n^2(\mathbf{r}')-1)$.
It is interesting to compare the validity of the Born and Rytov approximations.
Note that the Born approximation can be seen as a Taylor series approximation of the field $\psi(\mathbf{r},\varepsilon)$ in powers of ε, where ε is a measure of the inhomogeneity. The Rytov approximation can also be seen as a Taylor series approximation of $\log\psi(\mathbf{r},\varepsilon)$ in powers of ε. In our case, ε was the space-dependent variation n_δ from a constant refractive index.
We shall reproduce here the analysis by Keller (see Keller J.B. 1969 *'Accuracy and validity of the Born and Rytov approximations'*, J. Opt Soc. Am. 59, 1003-04) and consider the one-dimensional case of a wave travelling in a inhomogeneous medium given by

$$\psi(x,\varepsilon) = e^{ik(\varepsilon)x}\ , \tag{3.20}$$

and assume that $k(\varepsilon)$ is analytic in ε for $|\varepsilon|$ sufficiently small, so that it can be expanded in a power series in ε with coefficients $\mathbf{k}_j$:

$$k(\varepsilon) = \sum_{j=0}^{\infty} k_j\varepsilon^j\ . \tag{3.21}$$

3.2 Rytov Approximation

The Born expansion gives

$$\psi(x,\varepsilon) = e^{ik_0x}\sum_{s=0}^{\infty}\varepsilon^s\sum_{l=0}^{s}\frac{(ix)^l}{l!}\sum_{j_1+\cdots+j_l=s}k_{j_1}\cdots k_{j_l} \tag{3.22}$$

The nth Born approximation $\psi_B^{(n)}(x,\varepsilon)$ is the sum of the first $n+1$ terms in the expression above:

$$\psi_B^{(n)}(x,\varepsilon) = e^{ik_0x}\sum_{s=0}^{n}\varepsilon^s\sum_{l=0}^{s}\frac{(ix)^l}{l!}\sum_{j_1+\cdots+j_l=s}k_{j_1}\cdots k_{j_l} \tag{3.23}$$

The Rytov expansion gives

$$\psi(x,\varepsilon) = e^{ik(\sum_{j=0}^{\infty}k_j\varepsilon^j)}\ , \tag{3.24}$$

and the nth Rytov approximation $\psi_R^{(n)}(x,\varepsilon)$ is obtained by taking the first $n+1$ terms in the sum in the exponent:

$$\psi(x,\varepsilon) = e^{ik(\sum_{j=0}^{n}k_j\varepsilon^j)}\ . \tag{3.25}$$

The size of the error of the nth Born approximation $\psi - \psi_B^{(n)}$ for small ε and large $|x|$ can be found by examining the coefficient of ε^{n+1} in (3.22). That coefficient contains a term proportional to x^{n+1}. So

$$\psi - \psi_B^{(n)} = e^{ik_0x}\mathrm{O}(\varepsilon^{n+1}x^{n+1})\ . \tag{3.26}$$

Dividing this by ψ, and noting that ψ differs from e^{ik_0x} by terms of the order ε, we obtain for the relative error:

$$\frac{\psi - \psi_B^{(n)}}{\psi} = \mathrm{O}(\varepsilon^{n+1}x^{n+1})\ . \tag{3.27}$$

The error for the nth Rytov approximation $\psi - \psi_R^{(n)}$ is:

$$\begin{aligned} psi - \psi_R^{(n)} &= e^{ik(\sum_{j=0}^{\infty}k_j\varepsilon^j)} - e^{ik(\sum_{j=0}^{n}k_j\varepsilon^j)} \\ &= \psi\left(1 - e^{-ik(\sum_{j=n+1}^{\infty}k_j\varepsilon^j)}\right) \\ &= \psi\mathrm{O}(\varepsilon^{n+1}x) \end{aligned}$$

Dividing this by ψ gives for the relative error

$$\frac{\psi - \psi_R^{(n)}}{\psi} = \mathrm{O}(\varepsilon^{n+1}x)\ . \tag{3.28}$$

We can see then that the relative errors of the Born and the Rytov approximation are of the same order in the inhomogeneity parameter ε. However, the expressions obtained for the relative errors also show that they vary in a very different way as functions of x. For a single plane wave, the nth Rytov approximation is valid over a much larger range than is the nth Born approximation, however this advantage is lost for fields containing more than one wave, where the Rytov method must be applied to each wave separately and not to the total field ψ.

3.3 WKB Method

The WKB method (named after Wentzel, Kramers and Brioullin) is similar in concept to the Rytov approximation, since in this case also the field is assumed to have exponential form, and the exponent (phase) is approximated by a perturbation series. It is a method of obtaining approximate solutions to equations of the form

$$\frac{d^2\psi}{dx^2} + q(x)\psi = 0 \ , \tag{3.29}$$

where $q(x)$ is a slowly varying function of x. We derive this here for the one-dimensional case to keep the calculations simple. The generalization to higher dimensions can be found in Bremmer H and Lee SW 1984 *'Propagation of a geometrical field in an isotropic inhomogeneous medium' Radio Science* 19, 243-57. We shall take here $q(x) = k^2(x)$. If $q(x)$ were constant, then equation (3.29) would have solutions of the form:

$$\psi(x) = ae^{i\phi(x)} \ , \tag{3.30}$$

with a constant and $\phi(x) = \mp kx$. If k^2 varies slowly and the solution is written in the form

$$\psi(x) = a(x)e^{i\phi(x)} \ , \tag{3.31}$$

it is reasonable to expect $\phi(x)$ to vary rapidly and $a(x)$ to vary slowly. Inserting (3.31) into the Helmholtz equation (3.29) gives:

$$\frac{d^2a(x)}{dx^2} + 2i\frac{dk^2(x)}{dx}\frac{d\phi(x)}{dx} + ia(x)\frac{d^2\phi(x)}{dx^2} - a(x)\left(\frac{d\phi(x)}{dx}\right)^2 + k^2(x)a(x) = 0 \tag{3.32}$$

We now choose $\phi(x)$ such that

$$\left(\frac{d\phi(x)}{dx}\right)^2 - k^2(x) = 0 \ , \tag{3.33}$$

i.e. such that

$$\phi(x) = \mp \int^x k(x')dx' \ . \tag{3.34}$$

Since we made the assumption that $a(x)$ vary slowly, while $\phi(x)$ varies rapidly, we can neglect $\ddot{a}$ in comparison to $a\dot{\phi}$. Therefore equation (3.32)reduces to:

$$2\frac{\dot{a}}{a} + \frac{\ddot{\phi}}{\dot{\phi}} = 0 \ , \tag{3.35}$$

which can be integrated to give

$$a(x) = \propto \dot{\phi}^{-1/2} = \mp k^{-1/2} \ . \tag{3.36}$$

Hence, the approximate solution to (3.29) is

$$\psi(x) = \frac{1}{k^{1/2}} \left[A \exp\left(i \int_0^x k(x')dx' \right) + B \exp\left(-i \int_0^x k(x')dx' \right) \right] \ , \tag{3.37}$$

where A and B are arbitrary constants. It is apparent from the approximation made earlier (equation (3.33)), that this method is only valid for high frequencies. The nature of this approximation would have been even clearer if we had started with a 'trial solution' given by

$$\psi(x) = A e^{i\omega\tau(x)} \ , \tag{3.38}$$

with A constant, which would have given in the end the same $k^{-1/2}$ dependance of the amplitude of the solution. Briefly, we can see that using (3.38) in the Helmholtz equation gives

$$i\omega\ddot{\tau}(x) - \omega^2(\dot{\tau}(x))^2 + k^2(x) = 0 \ . \tag{3.39}$$

This can be solved using the perturbation series

$$\tau(x) = \tau_0(x) + \frac{1}{\omega}\tau_1(x) + \ldots, \text{ for } \omega \to \infty \ . \tag{3.40}$$

Note that $i\omega\ddot{\tau}_0(x) \ll k^2(x)$ as $\omega \to \infty$, because $k^2(x)$ is proportional to ω^2. Therefore, by substituting the perturbation series into the Helmholtz equation (3.33) and collecting leading terms, we have

$$\omega^2(\dot{\tau}_0(x))^2 = k^2(x) = \omega^2 s^2(x) \ , \tag{3.41}$$

where $s^2(x) = k/\omega$, which is independent of frequency. This differential equation can be solved to give

$$\tau_0(x) = \pm \int_{x_0}^x s(x')dx' + C_0 \ . \tag{3.42}$$

By collecting first order terms in (3.39), we have:

$$i\omega\ddot{\tau}_0(x) - 2\omega\dot{\tau}_0(x))\dot{\tau}_1(x)) = 0. \tag{3.43}$$

This gives

$$\tau_1(x) = \frac{i}{2}\ln\dot{\tau}_0(x) + C_1 = \frac{i}{2}\ln s(x) + C_{1\pm} \ . \tag{3.44}$$

By substituting the first two terms into the series, we find that the phase therefore is

$$\tau(x) = \pm \int_{x_0}^{x} s(x')dx' + \frac{i}{2\omega} \ln s(x) + C \ . \tag{3.45}$$

Using this result then we find the first order approximation to the field to be

$$\psi(x) \simeq \frac{1}{s^{1/2}} \left[A \exp\left(i\omega \int_0^x k(x')dx' \right) + B \exp\left(-i\omega \int_0^x k(x')dx' \right) \right] \ , \tag{3.46}$$

where A and B are arbitrary constants.

We can make some observations regarding the physical meaning of the WKB solution. It consists of a wave $\psi_+(x)$ travelling in the $+x$ direction, and a wave $\psi_-(x)$ travelling in the $-x$ direction The integrals in the exponents imply that the phase of a wave going from x_0 to x is proportional to the summation of all the phases gained locally at x' over the range from x_0 to x. In fact, this physical picture of the way the phase propagates is only valid if the multiple reflections of the wave can be neglected, and in general within the range of validity of this approximation.

Let us examine further the range of validity of (3.46). If we substitute the approximate solution $\psi_+(x)$ in the Helmholtz equation (3.29), we get

$$\frac{d^2\psi_+}{dx^2} + q(x)\psi_+ = f \neq 0 \ , \tag{3.47}$$

so we need to have

$$\mid f \mid \ll \mid q(x)\psi_+ \mid \ . \tag{3.48}$$

In our case, $q(x)$ is related to the refractive index by $q(x) = k^2(x) = k_0^2 n^2(x)$, so the condition (3.48) is:

$$\frac{1}{k_0^2} \left| \frac{3}{4n^4} \left(\frac{dn(x)}{dx} \right)^2 - \frac{1}{2n^3(x)} \frac{d^2n(x)}{dx^2} \right| \ll 1 \ . \tag{3.49}$$

The refractive index, therefore, must be a slowly varying function of x for the WKB approximation to hold. Also, we see that the solution becomes infinite and the approximation is not valid whenever $n(x) \simeq 0$ (or $s(x) \simeq 0$. The critical values x_0 at which this happens is called the *turning point*. It can be shown that, in the case of scattering at an interface between two media with different refractive index, it corresponds to the case when the angle of incidence is the critical angle θ_c (see section 2.1).

3.4 Parabolic Equation

Consider first a *scalar plane wave ψ in free space* (where we again assume and suppress a time-harmonic variation $e^{-i\omega t}$), with wavenumber k in a two-dimensional medium (x, z). As before x is horizontal and z is vertical. So ψ obeys the Helmholtz wave equation $(\nabla^2 + k^2)\,\psi = 0$. Suppose that ψ is propagating at a small angle α to the horizontal, say

$$\psi(x, z) = e^{ik(x\cos\alpha + z\sin\alpha)}\ . \tag{3.50}$$

Since $\sin\alpha$ is small we can approximate

$$\cos\alpha = \sqrt{1 - \sin^2\alpha} \cong 1 - \sin^2\alpha/2.$$

Now the fastest variation of ψ is close to the x direction, so define the 'slowly-varying' part E of ψ by

$$E = \psi e^{-ikx}$$

so that

$$E \cong e^{ik(-x\sin^2\alpha/2 + z\sin\alpha)}. \tag{3.51}$$

(E is also referred to as the reduced wave.) It then follows that

$$\frac{\partial E}{\partial x} = \frac{i}{2k}\frac{\partial^2 E}{\partial z^2}. \tag{3.52}$$

This is one form of the ***parabolic wave equation in free space***, and holds for any superposition of plane waves travelling at small angles to the horizontal. (Also referred to as the paraxial or forward scatter equation.)
It is straightforward to write the exact solution of (3.52) in terms of an initial value.
Let E be a field obeying (3.52). Define the Fourier transform of E with respect to z,

$$\hat{E}(x, \nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} E(x, z)e^{i\nu z}\, dz. \tag{3.53}$$

Taking the z-transform of (3.52) gives an equation for $\hat{E}$,

$$\frac{\partial \hat{E}}{\partial x} \ = \ -\frac{i\nu^2}{2k}\,\hat{E}. \tag{3.54}$$

This has solution (in terms of E at vertical plane $x = 0$)

$$\hat{E}(x, \nu) \ = \ e^{-i\nu^2 x/2k}\ \hat{E}(0, \nu). \tag{3.55}$$

Note that equation (3.52) can also be derived by substituting the form $E = \psi e^{ikx}$ into the Helmholtz wave equation for ψ, and neglecting terms of the form $\partial^2 E/\partial x^2$.

3.4 Parabolic Equation

We shall now consider the more general case of a *harmonic source in a refractive medium*. Let us consider a point source. It is natural then to use cylindrical coordinates (r, z, θ), and we shall restrict the problem to one where we assume azimuthal symmetry, so effectively again 2-dimensional, as the field is not dependent on θ. The Helmholtz equation is therefore

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 n^2 \psi = 0 \ , \tag{3.56}$$

where $k_0 = \omega/c_0$ is a reference wave number, and $n(r, z) = c_0/c(r, z)$ is the index of refraction of the medium.
Let us now rewrite the solution as

$$\psi(r, z) = \frac{u(r, z)}{\sqrt{r}} \ , \tag{3.57}$$

so we can go on to solve the Helmholtz equation for the wave $u(r, z)$, with the cylindrical spreading removed. In the far field, we obtain

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2 n^2 u = 0 \ . \tag{3.58}$$

If we now denote the operators appearing in this equation by

$$A = \frac{\partial}{\partial r} \ , \quad B = \sqrt{\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + n^2} \ , \tag{3.59}$$

we can factor equation (3.58) as

$$(A - ik_0 B)(A + ik_0 B)u - ik_0[A, B]u = 0 \ . \tag{3.60}$$

For a range-independent medium, where the refractive index does not depend on r, so $n \equiv n(z)$, A and B commute and the last term in (3.59) is zero. The remaining term corresponds to factorisation into one outgoing and one incoming wave component. Selecting only the outgoing wave component we obtain the one-way wave equation

$$Au = ik_0 Bu \tag{3.61}$$

or

$$\frac{\partial u}{\partial r} = ik_0 \left(\sqrt{\frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + n^2} \right) u \tag{3.62}$$

In order to use this equation in practice, a further approximation is necessary, to resolve the square root operator. If we write B as

$$B = \sqrt{1 + b} \ , \tag{3.63}$$

where

$$b = \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + n^2 - 1 \ , \tag{3.64}$$

then, if b is small, we can Taylor expand B and keep the first 2 terms to give the approximation

$$B \simeq 1 + \frac{b}{2} = 1 + \frac{1}{2k_0^2}\frac{\partial^2}{\partial z^2} + \frac{n^2-1}{2} \ . \tag{3.65}$$

Substituting this expression into (3.62) we obtain a parabolic equation for the 'full' wave in a refractive medium:

$$\frac{\partial u}{\partial r} = \frac{i}{2k_0}\frac{\partial^2 u}{\partial z^2} + \frac{ik_0}{2}(n^2+1)u \ . \tag{3.66}$$

If, as in the free space case, we again separate a 'slowly-varying' part E by defining

$$E = u(r,z)e^{-ikr} = \psi(r,z)\sqrt{r}e^{-ikr} \ , \tag{3.67}$$

then the Helmholtz equation for E is

$$\frac{\partial^2 E}{\partial r^2} + 2ik_0\frac{\partial E}{\partial r} - k_0^2 E + \frac{\partial^2 E}{\partial z^2} + k_0^2 n^2 E = 0 \ , \tag{3.68}$$

and the operator A in the factorisation is

$$A = \frac{\partial}{\partial r} + ik_0 \ , \tag{3.69}$$

leading to the more usual ***parabolic equation in a refractive medium***:

$$\frac{\partial E}{\partial r} = \frac{i}{2k_0}\frac{\partial^2 E}{\partial z^2} + \frac{ik_0}{2}(n^2-1)E \ . \tag{3.70}$$

It is seen here that the effect of the medium is contained in the second term on the right hand side. We may loosely think of the first term on the right as the *diffraction term*, and the second as the *scattering term*.

Other forms of the parabolic wave equation can be obtained by using different approximations for the square root operator.

This parabolic wave equation can also be derived (rather non-rigorously), as follows.

Again restricting ourselves to 2 dimensions, by regarding the wave as equivalent to the far field of a cylindrically spreading wave in a 3-dimensional medium with cylindrical symmetry, we shall take as starting point the same Helmholtz equation (3.56). As we are in the far field of this wave, we can replace the range r by the horizontal coordinates x, and take z as the vertical coordinate. We then denote by E the slowly varying part of ϕ,

$$E(x,z) = \psi(x,z)\sqrt{x}e^{-ik_0x}. \tag{3.71}$$

By substituting (3.71) into (3.56), and neglecting

1. all terms $O\left(x^{-\frac{3}{2}}\right)$ and higher order, since we are in the far field,

2. the term $\frac{\partial^2 E}{\partial x^2}$, which corresponds to slow variation across wavefronts and can be assumed to be small,

we obtain again the parabolic equation

$$\frac{\partial E}{\partial x} = \frac{i}{2k}\frac{\partial^2 E}{\partial z^2} + \frac{ik}{2}(n^2 - 1)E. \tag{3.72}$$

4 Scattering from randomly rough surfaces

4.1 Rayleigh criterion

We considered in section 2.1 the scattering of plane waves from a flat boundary between two media in 2 dimensions. This is an idealized case where analytical solutions to the scattering problem are straightforward and well-known. All real surfaces are rough. The scattering problem will then depend on the 'roughness' of the surface, and exact analytical solutions will not be generally available. In this chapter we shall look at ways of characterizing the surface, and consider some approximate solutions.
Suppose then that a time-harmonic plane wave

$$\psi_i = \exp(ik[x\sin\theta - z\cos\theta])$$

is incident on a boundary which is now an irregular function of position. (We suppress above and in what follows the harmonic time dependence). We will assume here that the *surface normal* is well-defined and continuous everywhere along the boundary. One of the earliest treatments of the rough surface problem was by Rayleigh (1907), who considered the phase change due to height differences in the case when the wavelength is small compared with the horizontal scale of surface variation.

Calculating the phase difference $\Delta\phi$ between wavefronts along two specularly reflected rays as in the schematic diagram gives

$$\Delta\phi = 2k(h_2 - h_1)\cos\theta$$

where h_1, h_2 are the heights at the two points of incidence. The interference between these two rays depends on the magnitude of $\Delta\phi$ with respect to π. When the surface is nearly flat, $\Delta\phi \ll \pi$ and the two rays are in phase (so interfere constructively), but for large deviations we may have $\Delta\phi \sim \pi$, giving destructive interference. This lead to the so-called Rayleigh criterion for distinguishing different roughness scales, by which surfaces may be called 'rough' or 'smooth' according to whether $\Delta\phi$ greater than or less than $\pi/2$.

If this is averaged across the surface, then $(h_2 - h_1)$ may be replaced by the average r.m.s. surface height σ, which gives the surface r.m.s. deviation from a flat surface, and is defined by $\sigma^2 =< h^2(x) >$. The Rayleigh criterion for 'smoothness' is then expressed by

$$k\sigma \cos\theta < \frac{\pi}{4} \ . \tag{4.1}$$

The quantity $k\sigma \cos\theta$ is referred to as the ***Rayleigh parameter***. Note that this is dependent on angle of incidence, and implies that all surfaces become 'smooth' for low grazing angles. At optical wavelengths this is often reasonable, but is less true, for example, for typical radar wavelengths of 3cm or whenever the roughness length scale becomes comparable to a wavelength. In that case the Rayleigh criterion fails to take into account 'multiple scattering' effects such as shadowing and diffraction.

4.2 Surface Statistics

When we go on to the study of the Helmholtz integral equation, one of the main goals is to find dependence of *averaged* quantities on the statistics of the surface. We therefore require a few concepts and results for surface statistics and characterisation. (The necessary results are not extensive but some familiarity with them is essential in the manipulation of the statistical quantities which arise, see for example Papoulis [7].)

Let S be a continuous irregular boundary, varying about a plane at, say, $z = 0$. We will assume that S can be represented as a function $h(x)$ of x, so that we can model this as a continuous stochastic process. We can think of h as a member of a given *ensemble* of surfaces all having the same statistical nature. All averages $< h(x) >$ etc are averages over this ensemble. (The angled brackets denote ensemble averages.)

Main assumptions: A number of assumptions are usually made about the statistics of the rough surfaces. This is for analytical convenience, but in most cases the assumptions are physically reasonable.

(1) The mean surface is *flat*, i.e. $< h(x) >= constant$ for all x (so we can choose $< h >= 0$).

(2) The surface h is *statistically stationary in* x, i.e. all statistics are translationally invariant. Thus, in particular the autocorrelation function $< h(x)h(x+\xi) >$ is a function of the spatial separation ξ only, and is constant in x.

(3) Surface heights are often assumed to be *normally distributed* (also referred to as Gaussian distributed, or simply as normal), i.e. they have

probability density function

$$f(h) = \frac{1}{\sigma\sqrt{2\pi}} e^{-h^2/2\sigma^2} \ . \tag{4.2}$$

For normal random variables we have the following:
If h is normal, then so is $h(x_1) + h(x_2)$, and $\int h(x)dx$ over any interval.
All the one-point statistics are determined by the mean $< h >$ and variance $< h^2 >$. For example we have $< h^{2n+1}(x) >= 0$ for all n, and

$$< h^4(x) >= 3\sigma^2 < h^2 > \tag{4.3}$$

This can be seen by writing

$$< h^n >= \int h'^n f(h')dh'$$

and integrating by parts, noting that in the case where $f(h)$ is Gaussian $hf(h) = -\sigma^2 \frac{d}{dh}(f(h))$.
The assumption of normal distributed heights is often physically reasonable; many rough surfaces arise as the result of a large number of independent random events ad are therefore normal by the Central Limit Theorem. However, it is wrong for important cases such as the sea surface. (The sea typically has sharper peaks than troughs, so the height distribution is not symmetric about the mean, as would be required by the symmetry of the normal distribution about the origin.)
There are three main measures with which to characterise roughness:

(1) **r.m.s. height** $\sigma = \sqrt{< h^2(x) >}$ (since we assume $< h >= 0$).

(2) **Autocorrelation function** (a.c.f)

$$\rho(x_1, x_2) =< h(x_1)h(x_2) >$$

By stationarity we can write this as a function of *spatial separation* only:

$$\rho(\xi) =< h(x)h(x + \xi) >$$

(3) **Correlation length** L: This is defined as the value of separation ξ at which $\rho(\xi) = e^{-1}\rho(0)$. So large L corresponds to a slowly varying surface. Instead of L we often use the *mean slope*, $< |dh/dx| >$. Clearly, slope scales with $1/L$.

The most general of these measures is clearly the a.c.f. (2), since this determines both the correlation length and r.m.s. height. It provides information about the spatial variation of the surface height, but is not related to the distribution of surface heights. The a.c.f. can have various forms depending on the nature of the irregularities.
Examples:
(a) Gaussian a.c.f.: $\rho(\xi) = \sigma^2 e^{-\xi^2/L^2}$
(b) Fractal surface: $\rho(\xi) = \sigma^2 e^{-|\xi|/L}$
(c) Fourth order power law: $\rho(\xi) = \sigma^2(1 + |\xi|)e^{-|\xi|/a}$

Unlike (b), the functions (a) and (c) are smooth at the origin, i.e. $d\rho/d\xi = 0$ at $\xi = 0$. Thus 'under a microscope' a surface of this type would appear smooth. The autocorrelation function (c) often occurs in other contexts, such as turbulence. We can assume that ρ is an even function, and falls from its maximum σ at $\xi = 0$ to zero at large $|\xi|$.
We also need the **roughness spectrum** (or **power spectrum**), that is the Fourier transform of the a.c.f.:

$$S(\nu) = \int_{-\infty}^{\infty} \rho(\xi) e^{i\xi\nu} d\xi$$

Finally in this section, the scattering solutions we seek are functions of the rough surface, involving integrals and derivatives of h. We therefore often need to evaluate the statistics of such functions, so we need some basic properties or rules for averaging.

(1) If $F(x)$ is a *deterministic* function, and $A(h)$ is any functional of the surface h, then

$$\left\langle \int A(h(x))F(x)dx \right\rangle = \int \langle A(h(x)) \rangle F(x)dx$$

This follows by linearity of the integral.

(2) A function which sometimes arises is the average of the product of h and its slope:

$$\left\langle h(y)\frac{dh(x)}{dx} \right\rangle = \left.\frac{d\rho}{d\xi}\right|_{\xi=y-x}.$$

In order to prove (2), write

$$h(y)h'(x) = h(y) \lim_{\epsilon\to 0} \frac{1}{\epsilon}\left[h(x+\epsilon) - h(x)\right]$$

The result follows by averaging the right-hand-side and taking the average inside the limit sign.

Numerical generation of random surfaces

It is instructive in the manipulation of averages to consider how a continuous rough surface $h(x)$ may be simulated. The simplest method is to represent $h(x)$ as a sum of sinusoidal components as follows:
Suppose we wish to represent an example of a surface with a given a.c.f. $\rho(\xi)$. The basic steps are:
(1) Define $A(\nu) = \sqrt{B(\nu)}$ where B is the cosine transform of ρ,

$$B(\nu) = \frac{2}{\pi} \int_{-\infty}^{\infty} \rho(\xi) \cos(\xi\nu)\ d\xi.$$

(We can assume that $B(\nu)$ has compact support.)
(2) Choose some number N of equally-spaced frequencies $\nu_j = j\Delta\nu$, say, where N and ν_N are large enough to resolve the features of B adequately.
(3) Choose N *independent random phases* ϕ_j, uniformly in $[0, 2\pi)$.
(4) Define a function $h(x)$ by

$$h(x) = \sqrt{\Delta\nu}\ \sum_{n=1}^{N} A_n \sin(\nu_n x + \phi_n),$$

where $A_n = A(\nu_n)$. Then h is a continuous function of x with the required statistics, as we can show. The random part of this definition is in the choice of random phases (3). Each different set of phases gives rise to a new realisation of a random process h, and averages can therefore be taken over this ensemble.
First, it is easy to check that $< h >= 0$, and for large N the values $h(x)$ are normally distributed by the central limit theorem. To calculate the a.c.f. of h, first write $x_n = \nu_n x + \phi_n$, and $y_n = \nu_n y + \phi_n$. Then since ϕ_n is uniform in $[0, 2\pi)$, it is easy to show for example that

$$\begin{aligned}
< \sin x_n > &= 0 \\
< \sin x_n \cos x_n > &= 0 \\
< \sin^2 x_n > &= 1/2 \\
< \sin x_n \sin y_n > &= \frac{1}{2}\cos(\nu_n \xi)
\end{aligned}$$

where $\xi = y - x$.

So the a.c.f. can be written

$$\begin{aligned} < h(x)h(y) > &= \Delta\nu \sum_{m,n=1}^{N} A_m A_n < \sin x_n \sin y_m > \\ &= \Delta\nu \sum_{n=1}^{N} A_n^2 < \sin x_n \sin y_n > \\ &= \frac{\Delta\nu}{2} \sum_{n=1}^{N} A_n^2 \cos(\nu_n \xi) \\ &\cong \int_{-\infty}^{\infty} B(\nu) \cos(\nu\xi)\ d\xi \\ &= \rho(\xi) \end{aligned}$$

as required. Here we have used the fact that $\sin x_n$ and $\sin y_m$ are independent.

4.3 Properties and Approximate Solutions of Scattering Equations

We will consider here the main methods used in solving the Helmholtz integral equations in the case of scattering from a rough surface, and the properties of the solutions.
Suppose that a plane wave

$$\psi_i(x,z) = e^{ik(x\sin\theta - z\cos\theta)}$$

impinges on a random rough surface $h(x)$. We will consider h to be a member of a statistical ensemble, which is stationary with respect to translation in x, with rms height $< h^2 >= \sigma^2$, autocorrelation function $\rho(\xi)$. We usually require:

the *scattered field* ψ_s;
the *coherent* (or mean) field $< \psi_s >$;
and the *field coherence function*

$$m(\xi) =< \psi_s(x)\psi*_s(y) > \text{ , where } \xi = y - x \text{ ,}$$

so that $m(0)$ is the mean intensity of the scattered field. It is often most important to find the *angular spectrum* $|\hat{\psi}(\nu)|^2$ or its average $< |\hat{\psi}(\nu)|^2 >$, where

$$\hat{\psi}(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_s(x,0)e^{-i\nu x}\, dx \tag{4.4}$$

i.e. the Fourier transform of ψ_s along the horizontal mean plane, $z = 0$. Each Fourier component $\hat{\psi}(\nu)$ will be scattered away from the surface $z = 0$ as another plane wave

$$\hat{\psi}(\nu)e^{iqz}$$

satisfying the Helmholtz equation. This gives $q = \sqrt{k^2 - \nu^2}$, where we have taken the positive (or positive imaginary) root to ensure that the scattered field consists of outgoing waves.
The field at a point (x,z) in the medium can therefore be written

$$\psi_s(x,z) = \int_{-\infty}^{\infty} \hat{\psi}(\nu)e^{i(\nu x + qz)}d\nu \tag{4.5}$$

General properties:
We can state some general properties of these quantities.
(1)*Relation between* $m(\xi)$ *and* $|\hat{\psi}(\nu)|^2$:

4.3 Properties and Approximate Solutions of Scattering Equations

Consider the autocorrelation of $\hat{\psi}$. From (4.4) we obtain

$$\left\langle \hat{\psi}(\nu')\hat{\psi}^*(\nu)\right\rangle = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \langle \psi(x)\psi^*(y)\rangle\, e^{-i\nu x+i\nu' y}\; dx\; dy. \tag{4.6}$$

Make the changes of variables $\xi = (x-y)/2$, $Y = (x+y)/2$. This then becomes

$$\begin{aligned}\left\langle \hat{\psi}(\nu)\hat{\psi}^*(\nu')\right\rangle &= \frac{1}{\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} m(2\xi)e^{-i(\nu+\nu')\xi-i(\nu-\nu')Y}\; d\xi\; dY \\ = & \quad \frac{2}{\pi}\delta(\nu-\nu')\int_{-\infty}^{\infty} m(\xi)e^{-i\nu\xi}\; d\xi. \end{aligned} \tag{4.7}$$

This is just $2/\pi\delta(\nu-\nu')$ times the Fourier transform of $m(\xi)$. Notice the important corollary of this, that $<\hat{\psi}(\nu)\hat{\psi}(\nu')>= 0$ for $\nu \neq \nu'$.

(2) *Energy conservation*, i.e. the average energy flux across a boundary in one direction must equal the average energy flux at the same point in the opposite direction. The averaged energy flux in a direction n was derived in section 2.1, and is given by (equation (2.10)):

$$E(\psi, n) = -\frac{\rho\omega}{2}\mathrm{Im}\left\{\psi^*\frac{\partial\psi}{\partial n}\right\}\; . \tag{4.8}$$

For the incident plane wave then, the average energy flux in the direction $n = -z$ across some horizontal line is (equation (2.11)):

$$E(\psi_i, n) = \frac{\rho\omega k\cos\theta}{2}\; , \tag{4.9}$$

and for the scattered field (4.5), the average energy flux in the direction $n = z$ is

$$E(\psi_s(x,z)) = \frac{\rho\omega k}{2}\int_{-\infty}^{\infty} |\hat{\psi}(\nu)|^2 q d\nu \tag{4.10}$$

So energy conservation implies

$$\cos\theta = \int_{-\infty}^{\infty} q|\hat{\psi}(\nu)|^2\; d\nu \tag{4.11}$$

where $q = \sqrt{k^2-\nu^2}$.

(3) *The mean field is specular*, i.e.

$$< \psi_s(x,z) >= R_e(\theta)\; e^{ik[x\sin\theta + z\cos\theta]} \tag{4.12}$$

where the (generally unknown) constant R_e is an 'effective reflection coefficient' which depends on the angle and the surface statistics. This result is a generalised form of Snell's law, and the mean transmitted field can be written similarly as a plane wave at the Snell's law angle. (Correspondingly, the mean spectrum $< \hat{\psi} >$ consists of a single delta-function peak.) The result follows from the assumption that the rough surface is statistically stationary. A corollary of this is that the mean of the full complex field shows no backscatter, or indeed any scatter outside the specular direction. This may initially surprising, but note that it does *not* apply to the mean amplitude $< |\hat{\psi}| >$ or the mean intensity or energy.

We now consider the two simplifying regimes of small surface height or small slope which allow approximate analytical solutions to be found.

(a) Small surface height $k\sigma \ll 1$:
In this case **perturbation theory** can be applied. The method is essentially to expand the functions appearing in the problem to form a simpler boundary problem on the mean plane, i.e. on $z =< h(x) >= 0$.
We seek the solution for the scattered field ψ_s and its mean $< \psi_s >$. Suppose that the surface obeys the Dirichlet condition, $\psi(x,h) = 0$. We proceed as follows:

(1) Expand the boundary condition to order h. Thus we obtain

$$\psi_i(x,0) + \psi_s(x,0) + h(x)\left(\frac{\partial \psi_i}{\partial z} + \frac{\partial \psi_s}{\partial z}\right) = 0 \;\; + O(h^2) \tag{4.13}$$

using $\psi = psi_i + \psi_s$. Here and below, unless specified otherwise, the functions are to be evaluated on the mean plane $z = 0$.

(2) Next, assume that the scattered field everywhere can be expanded in powers of kh, say

$$\psi_s(x,z) = \psi_0(x,z) + \psi_1(x,z) + \psi_2(x,z) + ... \tag{4.14}$$

where ψ_n is of order $O(h^n)$ for all n, so that ψ_0 is the known, deterministic *flat surface* reflected field, and ψ_n is stochastic for $n \geq 1$ since it depends on the specific choice of surface $h(x)$.

(3) Now truncate (4.14) at $O(h)$, substitute into (4.13), and neglect terms of order $O(h^2)$. This gives an *approximate boundary condition* which holds

on the mean plane

$$\psi_i + \psi_0 + \psi_1 + h(x)\left(\frac{\partial \psi_i}{\partial z} + \frac{\partial \psi_0}{\partial z}\right) = 0 \tag{4.15}$$

where again all functions are evaluated at points $(x, 0)$. In this equation the third term ψ_1 is the only unknown component, since the remaining functions are the zero order (flat surface) forms, so we have an explicit approximation to the solution along the mean plane.
The first two terms in (4.15) cancel, since they represent the total field which would exist in the case of a flat surface, which vanishes by the Dirichlet boundary condition. We can now equate terms of equal order. Equating $O(h)$ (first order) terms gives

$$\psi_1 = -h(x)\left.\frac{\partial(\psi_i + \psi_0)}{\partial z}\right|_{(x,0)}$$

which gives

$$\psi_1(x, 0) = -2h(x)\frac{\partial \psi_i}{\partial z}. \tag{4.16}$$

This solves for ψ_1 explicitly on the mean plane. From this we can obtain the scattered field everywhere to $O(h)$, using $\psi_s = \psi_0 + \psi_1 + O(h^2)$. Once ψ_1 is known on any plane we can split it into Fourier components, and propagate these outwards (using radiation conditions to determine the direction):
Consider in particular the case of an incident plane wave, $\psi_i = e^{ik(x\sin\theta - z\cos\theta)}$. We then have

$$\psi_1(x, 0) = -2h(x) ik\cos\theta \; e^{ikx\sin\theta}. \tag{4.17}$$

Denote by $\hat{h}$ the transform of h,

$$\hat{h}(\nu) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x)e^{-i\nu x}\, dx \; ,$$

then, from (4.4) and (4.17) we get

$$\hat{\psi}(\nu) \;=\; \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi_s(x, 0)e^{-i\nu x}\, dx = -ik\frac{\cos\theta}{\pi}\hat{h}(\nu - k\sin\theta) \; , \tag{4.18}$$

so that

$$\psi_1(x, z) = -ik\frac{\cos\theta}{\pi}\int_{-\infty}^{\infty} \hat{h}(\nu - k\sin\theta)\; e^{i(\nu x + qz)}\, d\nu \tag{4.19}$$

where as before $q = \sqrt{k^2 - \nu^2}$.

4.3 Properties and Approximate Solutions of Scattering Equations

Averaging:
The dependence of the field on the surface is now clear to first order in surface height. Taking the average of (4.19) immediately gives the mean of this perturbation as

$$< \psi_1 >= 0$$

everywhere, since $< h(x) >= 0$, so that *first order perturbation theory predicts no change in the coherent field.* (Equivalently, the effective reflection coefficient is the same to first order as the flat surface coefficient.) Although we have examined the Dirichlet condition it holds for arbitrary boundary conditions since the first order term is always linear in the boundary itself.

Angular spectrum:
Now consider the angular spectrum to find the scattered energy. For a plane wave incident at angle θ on a given surface, the far-field intensity is given by $I_\theta(\nu) = |\hat{\psi}(\nu)|^2$, so from (4.4), (4.17) we have

$$\left\langle |\hat{\psi}(\nu)|^2 \right\rangle = \left\langle \frac{k^2 \cos^2\theta}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x) h(x') e^{i(k\sin\theta - \nu)(x-x')} \, dx' \, dx \right\rangle . \tag{4.20}$$

Making the change of variables $\xi = (x - x')$, $X = (x + x')$, this becomes

$$\begin{aligned} \left\langle |\hat{\psi}(\nu)|^2 \right\rangle &= \frac{k^2 \cos^2\theta}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\xi) e^{i(k\sin\theta - \nu)\xi} \, d\xi \, dX \\ &= 2\frac{\delta(\nu)}{\pi} \, k^2 \cos^2\theta \; S(k\sin\theta - \nu) \end{aligned} \tag{4.21}$$

where S is again the surface spectrum and δ is the delta-function.

(b) Small surface slope:
We have been dealing with approximate solution in the case of small surface height. Now suppose that the *surface slopes are small*, i.e. $< |dh/dx| > \ll 1$. This approximation is used with the integral form of the wave equation (1.84), so the scattered field at $\mathbf{r}$ is given by

$$\psi_{sc}(\mathbf{r}) = \int_S \psi(\mathbf{r}_0)\frac{\partial G(\mathbf{r},\mathbf{r}_0)}{\partial n} - G(\mathbf{r},\mathbf{r}_0)\frac{\partial \psi}{\partial n}(\mathbf{r}_0)d\mathbf{r}_0 \ , \tag{4.22}$$

where $\mathbf{r}_0$ is on the surface and ψ and $\partial\psi/\partial n$ are unknown. We note that the use of this integral form implies integration over a closed surface, so will introduce errors (due to the edges) when the surface is not infinite.
The unknowns are approximated by using the **Kirchhoff approximation** (sometimes referred to as the tangent plane, or the geometrical optics solution), which treats any point on the scattering surface as though it were part of an infinite plane, parallel to the local surface tangent. We make the following assumptions:
(1) that the surface can be treated as 'locally flat';
(2) and that the incoming field at each point is just ψ_i.
The second assumption neglects multiple scattering, which can give rise to secondary illumination of any point on the surface.
Consider for simplicity the Dirichlet boundary condition, so that we are solving the integral equation

$$\psi_{sc}(\mathbf{r}_s) = -\int_S G(\mathbf{r},\mathbf{r}_0)\frac{\partial \psi}{\partial n}(\mathbf{r}_0)d\mathbf{r}_0 \ . \tag{4.23}$$

Under the assumptions above, we can approximate $\partial\psi/\partial n$ at each point by the value it would take for a flat surface with slope dh/dx:

$$\frac{\partial \psi}{\partial n} \cong -2\frac{\partial \psi_i}{\partial n}. \tag{4.24}$$

This neglects curvature and shadowing by other parts of the surface. The field then becomes

$$\psi_s(\mathbf{r}) = 2\int G(\mathbf{r},\mathbf{r}_0)\frac{\partial \psi_i}{\partial n}(\mathbf{r}_0)d\mathbf{r}_0. \tag{4.25}$$

Similar formulae are easily obtained for Neumann condition and more generally an interface between two media.
When the surface is not perfectly reflecting, the normal derivative of the field at the surface will be given by

$$\frac{\partial \psi}{\partial n} \cong (1 - R(\mathbf{r}_0))\frac{\partial \psi_i}{\partial n} \ , \tag{4.26}$$

where $R(\mathbf{r}_0)$ is the flat surface reflection coefficient; and the field at the surface by:

$$\psi \cong (1 + R(\mathbf{r}_0))\psi_i \ . \tag{4.27}$$

If we further consider the *far-field approximation*, we can approximate the argument of the free space Green's function, $k|\mathbf{r} - \mathbf{r}_0|$ by

$$k|\mathbf{r} - \mathbf{r}_0| \cong kr - k\hat{\mathbf{r}} \cdot \mathbf{r}_0 \ , \tag{4.28}$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of observation $\mathbf{r}$. The derivative of the Green's function can then be approximated by

$$\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} \cong -\frac{ie^{ikr}}{4\pi r}(\mathbf{n} \cdot \mathbf{k}_{sc})e^{-i\mathbf{k}_{sc}\cdot\mathbf{r}_0} \ , \tag{4.29}$$

where $\mathbf{k}_{sc} = k\hat{\mathbf{r}}$ is the wavevector of the scattered wave. Using these approximations in equation (4.22), we obtain for the scattered field

$$\psi_{sc}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r}\int_S ((R\mathbf{k}^- - \mathbf{k}^+) \cdot \mathbf{n})e^{-i\mathbf{k}^-\cdot\mathbf{r}_0} d\mathbf{r}_0 \ , \tag{4.30}$$

where

$$\begin{aligned} \mathbf{k}^- &= \mathbf{k}_i - \mathbf{k}_{sc} \\ \mathbf{k}^+ &= \mathbf{k}_i + \mathbf{k}_{sc} \ . \end{aligned}$$

If θ_1 is the angle of incidence (measured from the normal), and θ_2 and θ_3 are, respectively, the angle of the scattered wave with the normal, and the angle of the scattered wave with the x-axis in the plane (x, y), then

$$\begin{aligned} \mathbf{k}_i &= k(\hat{\mathbf{x}}\sin\theta_1 - \hat{\mathbf{z}}\cos\theta_1) \\ \mathbf{k}_{sc} &= k(\hat{\mathbf{x}}\sin\theta_2\cos\theta_3 + \hat{\mathbf{y}}\sin\theta_2\sin\theta_3 + \hat{\mathbf{z}}\cos\theta_2) \ . \end{aligned}$$

We can now convert the integration in equation (4.30) to integration over the mean plane of the surface, S_M, by noting that an area element of the rough surface, $d\mathbf{r}_0$, projects onto the mean plane of an area element of the mean plane $d\mathbf{r}_M$, with the area elements related by

$$\mathbf{n}d\mathbf{r}_0 \cong \left(\hat{\mathbf{x}}\frac{\partial h}{\partial x_0} - \hat{\mathbf{y}}\frac{\partial h}{\partial y_0} + \mathbf{k}\right) d\mathbf{r}_M \ . \tag{4.31}$$

The scattered field can therefore be written in the general form

$$\psi_{sc}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r}\int_{S_M}\left(a\frac{\partial h}{\partial x_0} + b\frac{\partial h}{\partial y_0} - c\right) e^{ik(Ax_0+By_0+Ch(x_0,y_0))}dx_0dy_0 \ , \tag{4.32}$$

where

$$\begin{aligned} A &= \sin\theta_1 - \sin\theta_2\cos\theta_3 \\ B &= -\sin\theta_2\sin\theta_3 \\ C &= -(\cos\theta_1 + \cos\theta_2) \ ; \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} a &= \sin\theta_1(1-R) + \sin\theta_2\cos\theta_3(1+R) \\ b &= \sin\theta_2\sin\theta_3(1+R) \\ c &= \cos\theta_2(1+R) - \cos\theta_1(1-R) \ . \end{aligned} \tag{4.34}$$

Note that this approximation for the scattered field has been derived within the far-field approximation, and for an incident plane wave. In order to make analytical manipulations possible, further approximations are usually made. *Note: the following part of Ch. 4 is non-examinable.*

In general, the reflection coefficient is a function of position on the surface. We shall assume instead that R *is constant*. With this approximation, and for $C \neq 0$, we can eliminate the terms involving partial derivatives of the surface by performing a partial integration. Carrying out the integration with the assumption of independent integration limits for x_0 and y_0, and taking the surface to be of finite extent, defined by $-X \leq x_0 \leq X$ and $-Y \leq y_0 \leq Y$, gives a scattered field of the form

$$\psi_{sc}(\mathbf{r}) = -\frac{ie^{ikr}}{4\pi r}2F(\theta_1,\theta_2,\theta_3)\int_{S_M} e^{ik\phi(x_0,y_0)}dx_0dy_0 + \psi_e \ , \tag{4.35}$$

where the phase function $\phi(x_0, y_0)$ is

$$\phi(x_0,y_0) = Ax_0 + By_0 + Ch(x_0,y_0) \ , \tag{4.36}$$

the angular factor $F(\theta_1, \theta_2, \theta_3)$ is

$$F(\theta_1, \theta_2, \theta_3) = \frac{1}{2}\left(\frac{Aa}{C} + \frac{Bb}{C} + c\right) \ , \tag{4.37}$$

and the term ψ_e is given by

$$\begin{aligned}\psi_e(\mathbf{r}) &= -\frac{ie^{ikr}}{4\pi r}\left[\frac{ia}{kC}\int\left(e^{ik\phi(X,y_0)} - e^{ik\phi(-X,y_0)}\right)dy_0\right.\\ &+ \left.\frac{ib}{kC}\int\left(e^{ik\phi(x_0,Y)} - e^{ik\phi(x_0,-Y)}\right)dx_0\right]\end{aligned} \tag{4.38}$$

In the above approximation the angular factor depends on the boundary conditions. The term ψ_e is often referred to as 'edge effects', since it involves the values of the phase function at the surface edges.

We can now calculate average quantities of the scattered field, when $h(x, y)$ is a random surface with some probability density $f(h)$. The average of the scattered field, i.e. the **coherent field** is given by

$$\psi_{sc}(\mathbf{r}) = -\frac{ie^{ikr}}{4\pi r}2F\int_{S_M}\int_{-\infty}^{\infty} e^{ik\phi(x_0,y_0)}f(h)dhdx_0dy_0 \ . \tag{4.39}$$

Assuming stationarity, and using the explicit expression for the phase function given by equation (4.36), we obtain

$$\psi_{sc}(\mathbf{r}) = -\frac{ie^{ikr}}{4\pi r}2F\hat{f}(kC)\int_{S_M} e^{ik(Ax_0+By_0}dx_0dy_0 \ , \tag{4.40}$$

where $\hat{f}(kC)$ is the Fourier transform of the probability density function, with respect to the transform variable kC.
The average of the intensity, i.e. the **angular spectrum**, is given by

$$\left\langle|\psi_{sc}|^2\right\rangle = \langle\psi_{sc}\psi *_{sc}\rangle - \langle\psi_{sc}\rangle\langle\psi *_{sc}\rangle \ . \tag{4.41}$$

This expression is far more complicated than the equivalent one obtained in the 'small height' approximation, because the coherent field is now different from zero. Further approximations will be necessary to obtain an expression of practical use for the angular spectrum in the Kirchoff approximation.

5 Wave Propagation through Random Media

References:

A. Ishimaru, Wave Propagation and Scattering in Random Media

B.J. Uscinski, Elements of Wave Propagation in Random Media

Remarks

This section concerns waves scattered by randomness or irregularities in the medium through which they are propagating. In many situations the wave speed varies randomly, for example in the atmosphere or the ocean. Sometimes this variation may be highly localized, such as a patch of turbulent air (e.g. over a hot road) or bathroom glass. These effects cause *focusing*, somewhat like that of a lens, and produce regions of both high and low intensity. (Familiar examples include the twinkling of stars, or the pattern of light in a swimming pool.)

There are essentially two mechanisms which contribute to this:

(i) *diffraction* (distance effect): i.e. the evolution of an irregular wave beyond a fixed plane. This allows focusing of rays as in a lens even when the medium is homogeneous; and

(ii) *scattering*, i.e. the continuous evolution of phase with propagation due to extended irregularities, causing bending of rays.

In an extended medium these effects of course occur simultaneously. We will consider these mechanisms only for *weakly scattering media*. Roughly speaking, 'weak scattering' corresponds to small angles of scatter, so that a plane wave may become scattered into a narrow range of directions close to the original direction. This allows us to use the *parabolic wave equation*, derived in section 3.4, which was derived as a small angle approximation.

Throughout this section we will assume that the parabolic equation holds, and that there is a definite predominant direction of propagation (which can be taken to be horizontal). It will be helpful to have the exact solution of the parabolic wave equation in free space, equation (3.52), in terms of an initial value.

We shall first consider the case in which the random irregularities occur within a thin layer.

5.1 Propagation beyond a thin phase screen

Suppose that we have initially a plane wave $\psi = e^{ikx}$ of unit amplitude propagating horizontally, so that the *reduced wave*, i.e. ψe^{-ikx}, is just $E(x,z) \equiv 1$. Suppose that E encounters a thin vertical layer in the region $x \in [-\xi, 0]$, say, in which the wave speed $c(z)$ is slightly irregular. (This may represent for example a jet of hot air, or a turbulent layer.)

Denote the **refractive index** $n(z) = c_0/c(z)$, where c_0 is the background or free wave speed. Write

$$n(z) = 1 + w(z) \ , \tag{5.1}$$

where the function $w(z)$ is small: $w(z) \ll 1$. We will assume that $w(z)$ is a continuous random fluctuation, with mean zero, i.e. $< w(z) >= 0$ for all z, stationary in z, and normally distributed.

Initial effect: In the assumption of weak scattering and for a thin enough layer, the field will only suffer a phase change on going through the layer. If a wave has wavenumber k before entering the layer, the wavenumber in the layer will be given by $kn(z) = k + kw(z)$, and the reduced wave will acquire a phase

$$\phi(z) = k\xi w(z), \tag{5.2}$$

where ξ is the thickness of the layer.

Then E emerges from the layer with a pure phase change,

$$E(0, z) \ = \ e^{i\phi(z)} \tag{5.3}$$

Evolution of the field and the moment equations

A primary aim of the study of random media is to examine the evolution of the field E with distance beyond the layer and find its statistics. (There are many reasons for this requirement: For example in ocean acoustics one can almost never know the refractive index in detail, but statistical information can help overcome communications and navigational problems, or may be used for remote sensing of the environment. In other situations the measurement devices themselves may be detecting time or spatial averages.)

5.1 Propagation beyond a thin phase screen

We introduce here some of the basic quantities and ideas, that will be developed further for an extended random medium. One notion providing a powerful tool for the analysis of wave statistics is that of the *moment equations.* This is applicable particularly in the case of the extended random medium, but best introduced first for the simpler case of propagation beyond a phase screen. Suppose for example we wish to find the mean intensity of the field. For a given medium it will not be possible to obtain a general solution for the wavefield or its intensity as a function of position. However, it is found that some of the statistical moments, such as field autocorrelation, themselves obey evolution equations. These take a relatively simple form since the fluctuations in the medium have been 'averaged out', and they can be solved or their solutions approximated analytically. We shall first consider first and second moments.

Denote the **first moment** or mean field by

$$m_1(x) \;=\; \langle E(x,z)\rangle \, . \tag{5.4}$$

This is a function of x only, by stationarity.
Similarly the **second moment** (transverse autocorrelation) of the field is defined as

$$m_2(x,\eta) \;=\; \langle E(x,z)E^*(x,z+\eta)\rangle \tag{5.5}$$

so that the mean of the intensity $I(x,z) = |E|^2$ can be written $< I(x) >= m_2(x,0)$.
We also denote the transverse autocorrelation ρ of the layer ϕ by

$$\rho(\eta) =< \phi(z)\phi(z+\eta) > \tag{5.6}$$

with variance

$$\sigma^2 = \rho(0) \tag{5.7}$$

and the *power spectrum*

$$S(\nu) = \int_{-\infty}^{\infty} \rho(\eta)e^{i\nu\eta} \, . \tag{5.8}$$

The initial intensity (immediately beyond the layer) is unchanged, so that $< I(0,z) >\equiv 1$, and the initial mean field is

$$m_1(0) =< e^{i\phi(z)} >= e^{-\sigma^2/2} \, . \tag{5.9}$$

(This is exact for the normal distribution as assumed here, in which case the the probability density function of ϕ is

$$f(\phi) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\phi^2/2\sigma^2} \, .$$

and approximate in general. It can be obtained from the definition

$$< e^{i\phi} >= \int_{-\infty}^{\infty} e^{i\phi} f(\phi) d\phi \ , \tag{5.10}$$

or simply by expanding the exponential and averaging term by term,

$$< e^{i\phi} >= 1 + i < \phi > - < \phi^2 > /2 - i < \phi^3 > /3 + ...) \tag{5.11}$$

As the field evolves, the pure phase fluctuations which are imposed initially become converted to amplitude variations. (In terms of ray theory, this happens as the layer focuses or de-focuses the rays passing through it, and the intensity changes with the ray density.)
This can be quantified roughly as follows:
At a small distance x beyond the layer, we can take a Taylor expansion of the field $E(x, z)$ about $E(0, z) = e^{i\phi(x)}$, using (5.3) and the parabolic wave equation (3.52):

$$E(x, z) \cong \left[1 + \frac{i}{2k} x (i\phi'' - \phi'^2) \right] \ e^{i\phi} \ , \tag{5.12}$$

where the prime denotes derivative, $\phi' = d\phi/dz$ etc., so that

$$I(x, z) \cong 1 - \frac{x}{k}\phi'' + \frac{x^2}{4k^2}(\phi''^2 + \phi'^2) \tag{5.13}$$

neglecting higher powers of x. This describes the initial mechanism for the build-up of amplitude fluctuations across the wavefront. As mentioned above, however, we can form *evolution equations*, i.e. differential equations governing the behaviour of the moments. These can be solved to find the far-field. Although the first few moment equations are trivial in the case of propagation beyond a layer we give them here as an introduction to the concept.

Evolution of the first moment (mean field):
By equation (5.9) we can write

$$< \hat{E}(0, \nu) > = \int_{-\infty}^{\infty} m_1(0) \ e^{i\nu z} \ dz \ = \ \sqrt{2\pi} \ \delta(\nu) \ e^{-\sigma^2/2}.$$

Taking the average of (3.55) then gives

$$< \hat{E}(x, \nu) > = \ \sqrt{2\pi} \ \delta(\nu) \ e^{-i\nu^2 x/2k} \ e^{-\sigma^2/2},$$

so that (because of the delta function) $< \hat{E}(x, \nu) >=< \hat{E}(0, \nu) >$ for all z, i.e.

$$\frac{dm_1}{dx} \ = \ 0. \tag{5.14}$$

so that the mean field is unchanged with distance.

Evolution of the second moment (vertical correlation of field):
We are also interested in mean intensity $< I(x) >$. Although we cannot form an evolution equation for $< I(x) >$ itself, we can do so for $m_2(x, \eta)$ and obtain $< I >$ by solving and setting $\eta = 0$.
The *initial condition* for m_2 at $x = 0$ is given by

$$m_2(0, \eta) = \left\langle e^{i[\phi(z_1) - \phi(z_2)]} \right\rangle$$

where $\eta = z_1 - z_2$. Since ϕ is normally distributed, so is the difference $\phi(z_1) - \phi(z_2)$. The variance of this difference is

$$\left\langle [\phi(z_1) - \phi(z_2)]^2 \right\rangle = 2 \left[\sigma^2 - \rho(\eta) \right]$$

This gives the initial value

$$m_2(0, \eta) \ = \ e^{-[\sigma^2 - \rho(\eta)]}. \qquad (5.15)$$

Now consider the 'transform' moment M_2 defined by

$$M_2(x, \nu_1, \nu_2) = \left\langle \hat{E}(x, \nu_1) \hat{E}^*(x, \nu_2) \right\rangle .$$

By equation (3.54) we get

$$\frac{\partial M_2}{\partial x} \ = \ \frac{i}{2k} (\nu_2^2 - \nu_1^2) \ M_2 \qquad (5.16)$$

However we can write M_2 directly in terms of E, as

$$\begin{aligned} M_2(x, \nu_1, \nu_2) \ &= \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle E(x, z_1) E^*(x, z_2) \rangle \ e^{i(\nu_1 z_1 - \nu_2 z_2)} \ dz_1 \ dz_2 \\ &= \ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m_2(x, \eta) \ e^{i(\nu_1 - \nu_2)Y/2 - i(\nu_1 + \nu_2)\eta/2} \ d\eta \ dY \qquad (5.17) \end{aligned}$$

where we have made the change of variables $\eta = z_1 - z_2$, $Y = z_1 + z_2$. Evaluating the Y-integral in (5.17) gives

$$M_2(x, \nu_1, \nu_2) \ = \ \sqrt{2\pi} \ \delta(\nu_1 - \nu_2) \int_{-\infty}^{\infty} m_2(x, \eta) \ e^{i(\nu_1 + \nu_2)\eta/2} \ d\eta \qquad (5.18)$$

so that M_2 vanishes unless $\nu_1 = \nu_2$. Hence we see from equation (5.16) that M_2, and therefore m_2, does not evolve with x, i.e.

$$\frac{\partial M_2}{\partial x} \ = \ \frac{\partial m_2}{\partial x} \ = \ 0. \qquad (5.19)$$

In particular the mean intensity remains constant. (It will be seen later that this no longer holds for an extended random medium.) We therefore need to go to higher moments to describe the intensity fluctuations which the eye and most 'square law' detectors observe in waves propagating through an irregular layer. Before doing that, we shall consider the evolution of the first and second moments in an extended random medium.

5.2 Propagation in an extended random medium

Consider now the second mechanism which can produce field fluctuations, that of extended refractive index irregularities. This is common in many situations, e.g. underwater acoustic, or atmospheric radio wave propagation. (Apart from any random irregularities there is often an underlying profile; for example the ocean sound channel which causes upward refraction of ray paths, confining sound to a region near the surface. This will not be treated here.)

Consider again a 2-dimensional medium (x, z) and a time-harmonic wave $\phi e^{i\omega t}$. Let $c(x, z)$ be the wave speed in the medium, and c_0 be the 'reference' or average wave speed. (We will take this as constant here although the actual profile may depend on depth.) Let $k = \omega/c_0$ be the corresponding wavenumber.

Denote the **refractive index** by $n(x, z) = c_0/c(x, z)$. We can write

$$n \;=\; 1 + n_d(z) + \mu W(x, z) \tag{5.20}$$

where n_d is the deterministic profile which, for example, allows for channelling, but which will be set to zero in the following derivation. μW is the random part, where W has been normalised, so that

$$< W >= 0, \quad < W^2 >= 1,$$

and therefore μ^2 is the variance of n. We will take W to be normally distributed, and *stationary in x and z*. We can then define the 2-dimensional autocorrelation function

$$\rho(\xi,\eta) \;=\; \mu^2 \langle W(x,z)\, W(x+\xi, z+\eta)\rangle \qquad (5.21)$$

so that $\rho(0,0) = \mu^2$. **Note** that ρ is assumed to decay to zero as $\xi \to \infty$ or $\eta \to \infty$. (This is reasonable unless there is an underlying periodicity in the medium.)
Further define the *horizontal and vertical length scales* H, L defined by

$$\rho(H,0) \;=\; \rho(0,L) \;=\; \mu^2 e^{-1}.$$

There are thus at least three measures affecting the scattering in different ways: μ^2, H, and L. We will look at their various effects on the field.

Weak scatter assumptions: We make the following assumptions, which correspond to different forms of weak scattering restrictions.

(1) *Small variation of refractive index*, $\mu^2 \ll 1$ (or equivalently $|n^2 - 1| \ll 1$).

(2) *Small angles of scatter*, expressed as

$$\lambda_0 \ll L$$

where λ_0 is the reference wavelength, $\lambda_0 = 2\pi/k_0$.

(3) *Weakly scattering medium*, i.e. the phase fluctuations imposed over a distance H are small,

$$k_0 \mu H \ll 1$$

Note: It will be seen below that 'stretching' the scale size H increases the scattering effect, whereas stretching the vertical scale L weakens it.

Under these weak scatter assumptions we shall be able to use the parabolic equation for an extended random medium, which was derived in section 3.4:

$$\frac{\partial E}{\partial x} \;=\; \frac{i}{2k}\frac{\partial^2 E}{\partial z^2} + \frac{ik}{2}(n^2(x) - 1)E. \qquad (5.22)$$

We will go on now to consider effects on the propagating field. First we investigate heuristically the effect of the horizontal length scale, and then will form and solve the basic moment equations.

5.2 Propagation in an extended random medium

Scattering effect of extended irregularities: For a given form of the medium W and its statistics, what is the effect of changes in the length scale H? For the moment we *ignore diffraction* and examine only the scattering term in (5.22).
Consider therefore a vertical layer consisting of the region $[x, x{+}d]$. Subdivide this into n thin subregions each of width $\Delta x = d/n$.

Each of these subregions, for $j = 1, ..., n$, imposes a normally-distributed phase change $\phi_j(z)$ with mean zero, whose variance is assumed to be given, say:

$$\langle \phi_j \rangle = 0, \ \langle \phi_j^2(z) \rangle = \delta^2. \tag{5.23}$$

So since we are ignoring diffraction the wave emerging at $x + d$ has the form

$$E(x + d, z) \ = \ E(x, z) \ e^{i\phi(z)} \tag{5.24}$$

where

$$\phi(z) \ = \ \sum_{i=1}^{n} \ \phi_i(z).$$

Now, since ϕ is normally distributed, the mean of this phase modification is

$$\left\langle e^{i\phi} \right\rangle \ = \ e^{-<\phi^2>/2} \tag{5.25}$$

so we want to examine the dependence of $< \phi^2 >$ on H. Consider two extreme cases:

(1) H small, say $H \leq \Delta x$: Then we can treat ϕ_i, ϕ_j as independent for all $i \neq j$, so that

$$\begin{aligned} \left\langle \phi^2 \right\rangle \ &= \ \left\langle \left(\sum_{i=1}^{n} \ \phi_i(z) \right)^2 \right\rangle \\ &= \ \sum_{i=1}^{n} \left\langle \phi_i^2(z) \right\rangle \\ &= \ n\delta^2 \end{aligned} \tag{5.26}$$

so that scattering scales linearly with n

(2) H large, say $H \gg d$: Then we can suppose that the medium at each depth z is approximately constant over the interval $[x, x+d]$,

$$\phi_i(z) = \phi_j(z) \text{ for all } i, j,$$

so that

$$< \phi^2 > = \left\langle [n\phi_1(z)]^2 \right\rangle = n^2 \, \delta^2. \tag{5.27}$$

Thus, increasing H magnifies the scattering effect of the medium.

Moment equations for an extended random medium

We now return to the problem of formulating and solving equations for the evolution of the moments, analogous to those for the thin layer.
Define again the **first moment**

$$m_1(x) \;\; = < E(x,z) > \tag{5.28}$$

where this quantity is again independent of z by the stationarity of W. Thus all z-derivatives $d^n m_1/dz^n$ vanish, so that *all effects on the mean field are due to the scattering term only* (in eq. (5.22)).
In order to derive the first moment equation, consider first the phase change $\phi(z)$ over a distance $d > H$ due to the scattering term only:

$$E(x+d, z) = E(x,z) \; e^{i\phi(z)} \tag{5.29}$$

where

$$\phi(z) = k_0 \mu \int_x^{x+d} W(x', z) \; dx'. \tag{5.30}$$

Square and average (5.30) to get

$$\begin{aligned} < \phi^2 > &= k_0^2 \mu^2 \int_x^{x+d} \int_x^{x+d} \langle W(x', z) W(x'', z) \rangle \; dx' \; dx'' \\ &= k_0^2 \mu^2 \int_x^{x+d} \int_x^{x+d} \rho(x' - x'', 0) \; dx' \; dx'' \,, \end{aligned}$$

where we have used the definition (5.21) for the transverse autocorrelation $\rho(x' - x'', 0)$. We now make the change of variables

$$\begin{aligned} \xi &= x' - x'' \\ X &= (x' + x'')/2 \end{aligned}$$

and use $d > H$ together with the fact that $\rho(X, 0) \sim 0$ for large X to obtain

$$\left\langle \phi^2 \right\rangle \;\cong\; k_0^2\mu^2 \int_0^d \int_{-\infty}^{\infty} \rho(\xi, 0)\; d\xi \; dX.$$

Therefore

$$\left\langle \phi^2 \right\rangle \;=\; k_0^2\mu^2 d\sigma_0 \tag{5.31}$$

where

$$\sigma_0 = \int_{-\infty}^{\infty} \rho(\xi, 0)\; d\xi.$$

Now, averaging (5.29) and using (5.31) gives

$$m_1(x+d) \equiv \langle E(x+d, z)\rangle \;\cong\; m_1(x)\; e^{-k_0^2\mu^2 d\sigma_0/2} \tag{5.32}$$

where we have made a further key assumption: *the field becomes independent of the medium*, due to the cumulative effect of scattering., i.e. for large x

$$\left\langle E(x, z)e^{i\phi(z)} \right\rangle \sim \langle E(x, z)\rangle \; \left\langle e^{i\phi(z)} \right\rangle .$$

It now follows directly from (5.32) that

$$m_1(x) \;=\; e^{(-k_0^2\mu^2\sigma_0/2)x}\; m_1(0). \tag{5.33}$$

Equivalently (or expanding $m_1(x+\xi)$ in ξ and comparing terms of $O(\xi)$ with a Taylor series) we can write

$$\frac{dm_1}{dx} \;=\; -(\frac{1}{2}k_0^2\mu^2\sigma_0)m_1 \; . \tag{5.34}$$

Thus $m_1(x)$ decays exponentially and is purely real.

Equation (5.34) for the evolution of the first moment due to the scattering term only, is also valid for a more general incident wave in 3 dimension, and amplitude different from 1, where the wave emerging from from a screen of width d is given by:

$$E(x+\xi, y, z) = E(x, y, z)e^{\phi(x+\xi,y,z)} \ ,$$

and we have

$$\frac{\partial m_1}{\partial x} \ = \ -(\frac{1}{2}k_0^2\mu^2\sigma_0)m_1 \ . \tag{5.35}$$

In general, we cannot disregard the 'diffraction' term, and we need to use

$$\frac{\partial E}{\partial x} = \frac{i}{2k_0}\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right)E + \frac{ik_0}{2}(n^2-1)E \ . \tag{5.36}$$

Therefore the equation for the first moment is

$$\frac{\partial m_1}{\partial x} \ = \ \frac{i}{2k_0}\left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2}\right)m_1 - (\frac{1}{2}k_0^2\mu^2\sigma_0)m_1 \ . \tag{5.37}$$

Since the medium is stationary, and therefore m_1 is independent of the transverse directions y, z, this reverts to equation (5.35)above.
For higher moments, the ∇^2 term must be retained, and the evolution equations can be solved by applying some small perturbations method, for example Born or Rytov, but only for small intensity fluctuations. Such solutions are of very limited use, since we know from experimental results and observations that even small randomness can give rise to very large intensity fluctuations.
It is possible to find a solution that allows for large intensity fluctuations by a *local* application of the method of small perturbation, and we shall derive moment equations and their solutions in this way. Conceptually then, using these moment equations to describe the evolution of the field is equivalent to using repeated applications of the Born approximation for successive (thin) screens.
Let us now consider the second moment

$$m_2 = < E_1(x, y_1, z_1)E_2^*(x, y_2, z_2) > \ , \tag{5.38}$$

where E_1 and E_2 represent E at two separate points in the same transverse plane at x.
Let us derive first the 'diffraction' term (or 'distance effect'). Consider

$$\frac{\partial}{\partial z}E_1E_2^* = E_2^*\frac{\partial E_1}{\partial z} + E_1^*\frac{\partial E_2}{\partial z} \ .$$

5.2 Propagation in an extended random medium

The diffraction term for the field at a single point E_i is

$$\frac{\partial E_i}{\partial x} = -\frac{i}{2k_0}\left(\frac{\partial^2}{\partial y_i^2} + \frac{\partial^2}{\partial z_i^2}\right) \equiv -\frac{i}{2k_0}\nabla_{Ti}^2 E_i \ . \tag{5.39}$$

Therefore

$$\frac{\partial}{\partial z}E_1E_2^* = -\frac{i}{2k_0}\left(E_2^*\nabla_{T1}^2 E_1 - E_1\nabla_{T2}^2 E_2^*\right) \ ,$$

and taking the ensemble average

$$\frac{\partial}{\partial x} < E_1E_2^* >= -\frac{i}{2k_0}\left(\nabla_{T1}^2 - \nabla_{T2}^2\right) < E_1E_2^* > \ . \tag{5.40}$$

We shall now consider the 'scattering' effect due to a screen of thickness d, so how the second moment $< E_1E_2^*(x) >$ evolves onto $< E_1E_2^*(x+d) >$. We have:

$$E_1E_2^*(x+d) = E_1E_2^*(x)e^{i[\phi(x+d,y_1,z_1)-\phi(x+d,y_2,z_2)]} \tag{5.41}$$

and

$$E_1E_2^*(x+d) = E_1E_2^*(x) - \frac{\partial}{\partial x} < E_1E_2^* > d \ , \tag{5.42}$$

so

$$E_1E_2^*(x+d) = E_1E_2^*(x)[1+i(\phi(y_1,z_1)-\phi(y_2,z_2))\frac{1}{2}(\phi(y_1,z_1)-\phi(y_2,z_2))^2+... \ , \tag{5.43}$$

where we have equated (5.41) and (5.42), and expanded the exponent. Taking the ensemble average, and remembering that $< \phi >= 0$, we have

$$\frac{\partial}{\partial x} < E_1E_2^* > d = -\frac{1}{2}\left\langle(\phi(y_1,z_1)-\phi(y_2,z_2))^2\right\rangle < E_1E_2^* > \ . \tag{5.44}$$

Now consider

$$\left\langle(\phi_1-\phi_2)^2\right\rangle = \left(< \phi_1^2 > -2 < \phi_1\phi_2 > + < \phi_2^2 >\right) \ , \tag{5.45}$$

where $\phi_1 = \phi(y_i, z_i)$ In the same way as we previously derived $< \phi^2 >$, (equation (5.31)), we can derive

$$\langle\phi_1\phi_2\rangle \ \cong \ k_0^2\mu^2\int_0^d\int_{-\infty}^{\infty}\rho(\xi,0)\ d\xi\ dX.$$

Therefore

$$\langle\phi_1\phi_2\rangle \ = \ k_0^2\mu^2 d\sigma_0 \tag{5.46}$$

where

$$\sigma_0 = \int_{-\infty}^{\infty}\rho(\xi,0)\ d\xi,$$

and ρ is the normalised autocorrelation function of the refractive index fluctuation:

$$\rho(\xi, \eta, \zeta) = \frac{1}{\mu^2} < W(x_1, y_1, z_1) W(x_2, y_2, z_2) > \ .$$

We can now use (5.46) in (5.45) to obtain

$$\left\langle (\phi_1 - \phi_2)^2 \right\rangle = 2k_0^2 \mu^2 d(\sigma(0,0) - \sigma(\eta, \zeta)) \ . \tag{5.47}$$

Therefore the evolution due to the scattering only is

$$\frac{\partial}{\partial x} < E_1 E_2^* >= -k_0^2 \mu^2 (\sigma_(0,0) - \sigma(\eta, \zeta)) < E_1 E_2^* > \tag{5.48}$$

Now, combining (5.48) and (5.40), we obtain the second moment equation

$$\frac{\partial m_2}{\partial x} = -\frac{i}{2k_0} \left(\nabla_{T1}^2 - \nabla_{T2}^2 \right) m_2 - k_0^2 \mu^2 (\sigma(0,0) - \sigma(\eta, \zeta)) m_2 \ . \tag{5.49}$$

Let us consider the ***fourth moment*** defined by

$$m_4 =< E_1 E_2^* E_3 E_4^* > \ . \tag{5.50}$$

We can derive an equation for the fourth moment in the same way as before:
The 'distance' effect:

$$\frac{\partial}{\partial x}(E_1 E_2^* E_3 E_4^*) = \frac{\partial E_1}{\partial x}(E_2^* E_3 E_4^*) + E_1 \frac{\partial E_2}{\partial x}(E_3 E_4^*) + E_1 E_2^* \frac{\partial E_3}{\partial x}(E_4^*) + E_1 E_2^* E_3^* \frac{\partial E_4}{\partial x} \ . \tag{5.51}$$

But

$$\frac{\partial E_i}{\partial z} = -\frac{i}{2k_0} \nabla_{Ti}^2 E_i \ ,$$

therefore:

$$\frac{\partial}{\partial x} < (E_1 E_2^* E_3 E_4^*) >= \frac{i}{2k_0} (\nabla_{T1}^2 - \nabla_{T3}^2 + \nabla_{T3}^2 - \nabla_{T4}^2) < (E_1 E_2^* E_3 E_4^*) > \ . \tag{5.52}$$

The 'scattering' effect:
We have

$$(E_1 E_2^* E_3 E_4^*)(x + d) = (E_1 E_2^* E_3 E_4^*)(x) e^{\phi_1 - \phi_2 + \phi_3 - \phi_4} \ , \tag{5.53}$$

and

$$(E_1 E_2^* E_3 E_4^*)(x + d) = (E_1 E_2^* E_3 E_4^*)(x) + \frac{\partial (E_1 E_2^* E_3 E_4^*)}{\partial x} d \ . \tag{5.54}$$

Equating (5.53) and (5.54), and expanding the exponent, we get

$$(E_1E_2^*E_3E_4^*)(x+d) = (E_1E_2^*E_3E_4^*)(x)[1+i(\phi_1-\phi_2+\phi_3-\phi_4)-\frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4)^2+... \quad (5.55)$$

So, truncating the expansion and taking the ensemble average, we have

$$\begin{aligned} \frac{\partial}{\partial x} & < E_1E_2^*E_3E_4^* > d = -\frac{1}{2}(\phi_1-\phi_2+\phi_3-\phi_4)^2 < E_1E_2^*E_3E_4^* > \\ = & -\frac{1}{2}\left(4 < \phi^4 > +2 < \phi_1\phi_3 > +2 < \phi_2\phi_4 > \right. \qquad (5.56) \\ - & \left. 2 < \phi_1\phi_2 > -2 < \phi_1\phi_4 > -2 < \phi_2\phi_3 > -2 < \phi_3\phi_4 > \right) < E_1E_2^*E_3E_4^* > , \end{aligned}$$

where we have used

$$< \phi_i\phi_j > = < \phi_j\phi_i >$$

and

$$< \phi_i\phi_i > = < \phi_i^2 > \ .$$

Now, proceeding as before, and remembering that (equation (5.46)),

$$< \phi_i\phi_j > = k_0^2\mu^2\sigma(\eta_j,\zeta_j)d \ , \qquad (5.57)$$

we can combine the 'distance' and 'scattering effects to obtain the **fourth moment equation**

$$\begin{aligned} \frac{\partial m_4}{\partial x} &= -\frac{i}{2k_0}(\nabla_{T1}^2-\nabla_{T3}^2+\nabla_{T3}^2-\nabla_{T4}^2)m_4 \\ &- k_0^2\mu^2\sigma(0,0)(2+\sigma_{13}+\sigma_{24}-\sigma_{12}-\sigma_{14}-\sigma_{23}-\sigma_{34})m_4 \ . \end{aligned} \qquad (5.58)$$

This is more usefully often written in the slightly different form:

$$\begin{aligned} \frac{\partial m_4}{\partial x} &= -\frac{i}{2k_0}(\nabla_{T1}^2-\nabla_{T3}^2+\nabla_{T3}^2-\nabla_{T4}^2)m_4 \\ &- \beta(2+f_{13}+f_{24}-f_{12}-f_{14}-f_{23}-f_{34})m_4 \ , \end{aligned} \qquad (5.59)$$

where

$$\beta = k_0^2\mu^2\sigma(0,0) \qquad (5.60)$$

and

$$f_{i,j} = \sigma(y_i-y_j, z_i-z_j)/\sigma(0,0) \ . \qquad (5.61)$$

The quantity β defined above is the so-called *attenuation coefficient*, which is a useful parameter of physical significance. β is of course related to the attenuation of the mean field $< E >$ by the medium, since

$$< E > = E_0e^{-\beta/2} \ ,$$

and to the 'unscattered' power $< E >^2$ by

$$< E >^2 = E_0 e^{-\beta} \ , \tag{5.62}$$

It has a further physical significance in terms of the mean free path of a photon in a random medium. Suppose the incident (electromagnetic) field is regarded as a unidirectional flux of photons. The incident field is attenuated exponentially like

$$e^{\mathbf{x}/x_m}$$

as it passes through the medium, where x_m is the mean free path of the photon in the medium. The number of photons in the unscattered flux is proportional to the unscattered power, so, comparing with (5.62), we see that β^{-1} may be interpreted as the mean free path of a photon in a random medium.

Let us now find the solutions for the second and fourth moment.

Solution of the second moment equation
It is convenient to use the set of variables

$$\begin{aligned} \xi = x_1 - x_2 \ , \ \eta = y_1 - y_2 \quad &, \quad \zeta = z_1 - z_2 \\ X = x_1 + x_2 \ , \ Y = y_1 + y_2 \quad &, \quad Z = z_1 + z_2 \end{aligned} \tag{5.63}$$

and to set

$$< E_1 E_2^* > e^{\beta x} = u(\beta x, \eta, \zeta) \ . \tag{5.64}$$

The equation for the second moment then can be written as

$$\frac{\partial u}{\partial(\beta x)} = -\frac{2i}{k_0 \beta}\left(\frac{\partial^2 u}{\partial Y \partial \eta} + \frac{\partial^2 u}{\partial Z \partial \zeta}\right) + \frac{\sigma(\eta,\zeta)}{\sigma(0,0)} u \ . \tag{5.65}$$

It is convenient to transform this equation using the transform pair

$$\begin{aligned} u(Y,\eta;Z,\zeta) &= \int\!\!\int \hat{u}(\eta,\zeta;\epsilon_1,\epsilon_2) e^{i(\epsilon_1 Y + \epsilon_2 Z)} d\epsilon_1 d\epsilon_2 \ , \\ \hat{u}(\eta,\zeta;\epsilon_1,\epsilon_2) &= \frac{1}{2\pi}\int\!\!\int u(Y,\eta;Z,\zeta) e^{-i(\epsilon_1 Y + \epsilon_2 Z)} dY dZ \ . \end{aligned}$$

to obtain

$$\frac{\partial \hat{u}}{\partial(\beta x)} = B_1 \frac{\partial \hat{u}}{\partial \eta} + C_1 \frac{\partial \hat{u}}{\partial \zeta} + \frac{\sigma(\eta,\zeta)}{\sigma(0,0)} \hat{u} \ , \tag{5.66}$$

where

$$B_1 = 2\epsilon_1/k\beta \ , \ C_1 = 2\epsilon_2/k\beta \ .$$

The general solution of (5.66)is

$$\hat{u} = \hat{u}_0(\eta+B_1\beta x, \zeta+C_1\beta x) \exp\left[\int_0^{\beta x} \frac{\sigma(\eta + B_1(\beta x - \beta x');\ \zeta + C_1(\beta x - \beta x'))}{\sigma(0,0)} d(\beta x')\right] , \tag{5.67}$$

where $\hat{u}_0$ is the solution of the transform equation (5.66) when $\sigma(\xi, \eta, \zeta) = 0$. The second moment then is given by the inverse transform, which, in our case where $Y = Z = 0$, reduces to

$$u(Y, \eta; Z, \zeta) = \int\int \hat{u} d\epsilon_1 d\epsilon_2 \ . \tag{5.68}$$

If the incident field is a plane wave with amplitude E_0 at $x = 0$ and propagating parallel to the x-direction, then

$$\hat{u}_0 = \delta(\epsilon_1)\delta(\epsilon_2) \tag{5.69}$$

and from (5.67) and the inverse transform we have:

$$\begin{aligned} u &= E_0^2 \exp\left[\int_0^{\beta x} \frac{\sigma(\eta;\zeta) d(\beta x')}{\sigma(0,0)}\right] \\ &= E_0^2 \exp\left[-\beta x \left(1 - \frac{\sigma(\eta;\zeta)}{\sigma(0,0)}\right)\right] \ . \end{aligned} \tag{5.70}$$

Solution of the fourth moment equation
We shall now seek a solution of equation (5.59):

$$\begin{aligned}\frac{\partial m_4}{\partial x} &= -\frac{i}{2k_0}(\nabla^2_{T1} - \nabla^2_{T3} + \nabla^2_{T3} - \nabla^2_{T4})m_4 \\ &- k_0^2\mu^2\sigma_0(2 + f_{13} + f_{24} - f_{12} - f_{14} - f_{23} - f_{34})m_4\ , \qquad (5.71)\end{aligned}$$

We shall follow similar steps to those used to find a solution of the second moment equation, so we shall first make an appropriate change of variables, then use Fourier transforms.
Denote by L the scale size of the inhomogeneities transverse to the direction of propagation x, and define a new variable, scaling x by the so-called 'Fresnel length' kL^2:

$$X = \frac{x}{kL^2} \qquad (5.72)$$

Introduce also the following scaled variables:

$$\begin{aligned}\zeta_a &= (z_1 - z_2 - z_3 + z_4)/2L \\ \zeta_b &= (z_1 + z_2 - z_3 - z_4)/2L \\ \zeta_c &= (z_1 - z_2 + z_3 - z_4)/2L \\ Z &= (z_1 + z_2 + z_3 + z_4)/L\end{aligned} \qquad (5.73)$$

and analogous ones relating in the y coordinate:

$$\begin{aligned}\eta_a &= (y_1 - y_2 - y_3 + y_4)/2L \\ \eta_b &= (y_1 + y_2 - y_3 - y_4)/2L \\ \eta_c &= (y_1 - y_2 + y_3 - y_4)/2L \\ Y &= (y_1 + y_2 + y_3 + y_4)/L\end{aligned} \qquad (5.74)$$

We shall also define the parameter

$$\Gamma = k^3\mu^2\sigma_0 L^2 \qquad (5.75)$$

For simplicity, we shall restrict the following to 2 dimensions, in the plane (x, z). It will be straightforward to extend the final result to include the y coordinate. In this case then, and with the new variables defined above, the fourth moment equation becomes

$$\frac{\partial m_4}{\partial X} = -i\left(\frac{\partial^2 m_4}{\partial\zeta_a\partial\zeta_2} + \frac{\partial^2 m_4}{\partial\zeta_c\partial Z}\right) - 2\Gamma(1 - g(\zeta_a, \zeta_b, \zeta_c))m_4\ , \qquad (5.76)$$

We shall now seek the solution in 2D, for a plane wave of unit amplitude normally incident onto the half-space $x > 0$. Equation (5.71) then simplifies further, since in this case the 4th moment is independent of the transverse direction Z, and all the field quantities in (5.76) can be written as functions of 2 new variables only. We have:

$$\frac{\partial m_4}{\partial X} = -i\frac{\partial^2 m_4}{\partial \zeta_a \partial \zeta_2} - 2\Gamma(1 - g(\zeta_a, \zeta_b))m_4 \ , \tag{5.77}$$

where now

$$g = f(\zeta_a) + f(\zeta_B) + -\frac{1}{2}f(\zeta_a + \zeta_b/2) - \frac{1}{2}f(\zeta_a - \zeta_b/2)$$

Similarly to the procedure followed to find the solution for the second moment, set

$$m_4 e^{2\Gamma X} = m \ ,$$

then use this in (5.77), and multiply the resulting equation by $e^{-2\Gamma X}$, to obtain

$$\frac{\partial m}{\partial X} = -i\frac{\partial^2 m}{\partial \zeta_a \partial \zeta_2} + 2\Gamma g m \ , \tag{5.78}$$

Again we transform this equation using Fourier transforms:

$$\begin{aligned} M &= \frac{1}{2\pi}\int\int m(\zeta_a, \zeta_b, X)e^{-i(\nu_a\zeta_a + \nu_b\zeta_b)}d\zeta_a d\zeta_b \ , \\ G &= \frac{1}{2\pi}\int\int g(\zeta_a, \zeta_b)e^{-i(\nu_a\zeta_a + \nu_b\zeta_b)}d\zeta_a d\zeta_b \ , \end{aligned}$$

and obtain

$$\frac{\partial M}{\partial X} = i\nu_a\nu_b M + 2\Gamma\int\int G(\nu_a - \nu_a^{'}, \nu_b - \nu_b^{'})M(\nu_a^{'}, \nu_b^{'}, X) \tag{5.79}$$

In order to solve this integrodifferential equation, we shall now represent M as a series:

$$M = \sum_{n=0}^{\infty} M_n \ , \tag{5.80}$$

where

$$M_0 = M(\nu_a, \nu_b, 0) = \delta(\nu_a)\delta(\nu_b) \ , \tag{5.81}$$

which we take as the initial condition for (5.79). Using now the series (5.80) in (5.80) gives:

$$\frac{\partial M_n}{\partial X} = i\nu_a\nu_b M_n + 2\Gamma\int\int G(\nu_a - \nu_a^{'}, \nu_b - \nu_b^{'})M_{n-1}(\nu_a^{'}, \nu_b^{'}, X) \tag{5.82}$$

We can solve (5.82) starting from M_1:

$$\frac{\partial M_n}{\partial X} = i\nu_a \nu_b M_n + 2\Gamma G(\nu_a, \nu_b) \;, \tag{5.83}$$

with initial condition $M_1(\nu_a, \nu_b, 0) = 0$. This has solution

$$M_1(\nu_a, \nu_b, X) = 2\Gamma \int_0^X G(\nu_a, \nu_b) e^{\nu_a \nu_b (X - X_1)} dX_1 \tag{5.84}$$

Now we can use M_1 to solve for M_2:

$$\frac{\partial M_2}{\partial X} = i\nu_a \nu_b M_2 + 2\Gamma \int\int G(\nu_a - \nu_a^{'}, \nu_b - \nu_b^{'}) M_1(\nu_a^{'}, \nu_b^{'}, X) d\nu_a^{'} d\nu_b^{'} \tag{5.85}$$

with initial condition $M_2(\nu_a, \nu_b, 0) = 0$. This has solution

$$\begin{aligned} M_2(\nu_a, \nu_b, X) &= \\ (2\Gamma)^2 & \int_0^X \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\nu_a - \nu_{a_1}, \nu_b - \nu_{b_1}) M_1(\nu_{a_1}, \nu_b^{'}, X_2) e^{i\nu_a \nu_b (X - X_2)} dX_2 d\nu_{a_1} d\nu_{b_1} \\ = (2\Gamma)^2 & \int_0^X \int_0^{X_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\nu_a - \nu_{a_1}, \nu_b - \nu_{b_1}) G(\nu_a - \nu_{a_1}, \nu_1 b, X_2) \times \\ & e^{i\nu_a \nu_b (X - X_2) + i\nu_{a_1} \nu_{b_1} (X_2 - X_1)} dX_2 d\nu_{a_1} d\nu_{b_1} \end{aligned}$$

and the nth term in the series is:

$$\begin{aligned} M_n(\nu_a, \nu_b, X) &= (2\Gamma)^n \int_0^X \int_0^{X_n} \ldots \int_0^{X_2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} G(\nu_{a_1}, \nu_{b_1}) \\ &\times \; G(\nu_{a_2} - \nu_{a_1}, \nu_{b_2} - \nu_{b_1}) \\ &\times \; G(\nu_{a_3} - \nu_{a_2}, \nu_{b_3} - \nu_{b_2}) \\ & . \\ & . \\ & . \\ &\times \; G(\nu_a - \nu_{a_{n-1}}, \nu_b - \nu_{b_{n-1}}) \\ &\times \; e^{i(\nu_{a_1} \nu_{b_1} (X_2 - X_1) + \nu_{a_2} \nu_{b_2} (X_3 - X_2) + \ldots + \nu_a \nu_b (X - X_n)} \\ &\times \; d\nu_{a_1} \ldots d\nu_{a_{n-1}} d\nu_{b_1} \ldots d\nu_{b_{n-1}} dX_1 \ldots dX_n \end{aligned}$$

Replacing all the terms G by their inverse transform and carrying out all the possible integrals gives:

$$\begin{aligned} M_n(\nu_a, \nu_b, X) &= \frac{(2\Gamma)^n}{(2\pi)^{2n+2}} \int_0^X \int_0^{X_n} \ldots \int_0^{X_2} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} \prod_{j=1}^{n} g(\zeta_{a_1}, \zeta_b + Q_j) \\ &\times \; e^{i\zeta_{a_j} (\nu_{a_j} - \nu_{a_{j-1}}) - i\nu_b \zeta_b} \\ &\times \; d\zeta_b d\zeta_{a_1} \ldots d\zeta_{a_n} \\ &\times \; d\nu_{a_1} \ldots d\nu_{a_{n-1}} dX_1 \ldots dX_n \;, \end{aligned}$$

where

$$Q_j = (\nu_a(X - X_n) + \nu_{a_{n-1}}(X_n - X_n) + ... + \nu_{a_j}(X_{j+1} - X_j)) \ . \tag{5.86}$$

It is useful to interpret the subscripts n as the number of times that the field is scattered, contributing to a given term in the series, M_n. The "single scatter" approximation then corresponds to taking just the first term in the series, which is

$$M_1(\nu_a, \nu_b, X) = 2\Gamma \int_0^X G((\nu_a, nu_b) e^{i(\nu_a \nu_b (X - X_1)} dX_1 \tag{5.87}$$

and we recover the Born approximation.

In order to obtain higher order terms in the sum, further approximations are necessary, which of course involve further errors, beyond those incurred in truncating the series.
Because of this, and because of the quite complicated analytical form of the higher order corrections, which do not readily yield physical insight, calculations are most usefully carried out using *numerical approximations*.
We can write the equation for the fourth moment (equation (5.77)) in terms of operators as:

$$\frac{\partial m_4}{\partial X} = (A(X) + B(\Gamma, X))\, m_4 \ , \tag{5.88}$$

where

$$A = {}_1 \frac{\partial^2}{\partial \zeta_a \partial \zeta_b} \ ; \ B = -2\Gamma(1 - g(\zeta_a, \zeta_b))m \ . \tag{5.89}$$

We can write $C = A + B$. Difficulties can arise in the numerical solution for C, particularly for large values of Γ, when semi-discretization leads to a stiff system of differential equations. Furthermore, since there are two transverse variables, the matrices which operate on this system are of order N^4, where N is the number of points in the discretization along each axis.
The formal solution of equation (5.88) over the range $(X, X + \Delta X)$ is

$$m_4(\zeta_a, \zeta_b, X + \Delta X) = e^{\int_X^{X+\Delta X} C(\Gamma, X')dX'} m_4(\zeta_a, \zeta_b, X) \ . \tag{5.90}$$

In the case of a plane wave, C does not vary with X, so

$$\int C dX = \Delta X C$$

and the exact formal solution is

$$m_4(\zeta_a, \zeta_b, X + \Delta X) = e^{\Delta X(A+B)} m_4(\zeta_a, \zeta_b, X) \tag{5.91}$$
$$= \left(1 + \Delta X(A + B) + \frac{(\Delta X)^2}{2}(A + B)^2 + ...\right) m_4(\zeta_a, \zeta_b, X).$$

We first approximate this with the 'operator splitting' solution given by

$$m_4(\zeta_a, \zeta_b, X + \Delta X) = e^{\Delta X A} e^{\Delta X B)} m_4(\zeta_a, \zeta_b, X) \tag{5.92}$$
$$= \left(1 + \Delta X A + \frac{(\Delta X)^2}{2} A^2 + ...\right)\left(1 + \Delta X B + \frac{(\Delta X)^2}{2} B^2 + ...\right) m_4(\zeta_a, \zeta_b, X).$$

This is exact only if the operators A and B commute. We can see, by expanding the terms in (5.91) and comparing with (5.92),that the step-wise error in (5.92) is $\mathrm{O}[(\Delta X)^3]$. However, the overall accuracy depends on the degree of commutativity between A and B in the strong operator topology or, in other words, on the quantity $\| (AB - BA) m_4 \|$. This quantity is indeed very small, so the method is very accurate.

This operator splitting can be applied when the irregularities in the medium have any given autocorrelation function with an outer scale, even if it is range-dependent.

The method is unconditionally stable and convergent, and can be applied even when there is strong scattering.

The method allows comparison of analytical and numerical intensity fluctuation spectra over a wide range of Γ and X.

The scintillation index

The scintillation index S_I^2 is the normalised variance (sometimes referred to as '*mean square*') of the intensity fluctuations:

$$S_I^2 = \frac{< I^2 > - < I >^2}{< I >^2} \ , \tag{5.93}$$

where the intensity I is given by

$$I = EE^* \ .$$

S_I^2 is a measure of the fluctuations of the received signal by most devices, and is of course a fourth moment of the field.

6 Electromagnetic scattering in layered media

We derived in chapter 1 the time-dependent wave equation for electromagnetic waves in free space, equation (1.39), and the corresponding Helmholtz equation for time-harmonic electromagnetic waves, equation (1.42). We shall now extend these results to a general medium, and apply them to typical scattering problems.

If the medium is *anisotropic*, then the permittivity ϵ and the permeability μ are 2nd rank tensors, because the medium has different propagation characteristics along different axes. If the medium is *inhomogeneous*, the permittivity and permeability are space-dependent. Therefore, for a general anisotropic, inhomogeneous medium, with a source induced by a current density $\mathbf{J}$, the second and third Maxwell equations are

$$\begin{aligned} \nabla \times \mathbf{E} &= -\frac{\partial}{\partial t}\bar{\mu}(x,y,z)\cdot\mathbf{H} && (6.1)\\ \nabla \times \mathbf{H} &= \frac{\partial}{\partial t}\bar{\epsilon}(x,y,z)\cdot\mathbf{E}+\mathbf{J} && (6.2) \end{aligned}$$

and in this case $\bar{\epsilon}(x,y,z)$ and $\bar{\mu}(x,y,z)$ do not commute with the ∇ operator. For time-harmonic waves, where the time dependence is given by $e^{i\omega t}$, then we shall have reduced (Helmholtz) wave equations given by:

$$\begin{aligned} \nabla \times \bar{\mu}^{-1}\cdot\nabla\times\mathbf{E}-\omega^2\bar{\epsilon}\cdot\mathbf{E} &= i\omega\mathbf{J} && (6.3)\\ \nabla \times \bar{\epsilon}^{-1}\cdot\nabla\times\mathbf{H}-\omega^2\bar{\mu}\cdot\mathbf{H} &= \nabla\times\bar{\epsilon}^{-1}\cdot\mathbf{J}\ . && (6.4) \end{aligned}$$

Where, to obtain the Helmholtz wave equation for $\mathbf{E}$, we have taken $\nabla\times\bar{\mu}^{-1}$ of (6.1) and used (6.2); and similarly to obtain the Helmholtz wave equation for $\mathbf{H}$, we have taken $\nabla\times\bar{\epsilon}^{-1}$ of (6.2) and used (6.1).

For an inhomogeneous, isotropic medium, these reduce to

$$\begin{aligned} \nabla \times \mu^{-1}\cdot\nabla\times\mathbf{E}-\omega^2\epsilon\mathbf{E} &= i\omega\mathbf{J} && (6.5)\\ \nabla \times \epsilon^{-1}\cdot\nabla\times\mathbf{H}-\omega^2\mu\mathbf{H} &= \nabla\epsilon^{-1}\mathbf{J}\ . && (6.6) \end{aligned}$$

If we have plane polarised waves, i.e. TE or TM waves, we can see that the wave propagation problem will always be described by a system of *coupled* vector equations in the case of an inhomogebeous anisotropic medium, whilst it will be possible in general for an isotropic medium to choose a coordinate system in which wave propagation is described by a system of six *uncoupled* scalar equations.

If the medium is anisotropic, but homogeneous, TE and TM waves are, in general, also decoupled. However, in the presence of a planar interface they

will be coupled to each other at the interface; an incident TE wave may generate both TE and TM waves, and propagation has to be treated by vector equations. This is also the case when a polarised wave travelling in a homogeneous isotropic medium is scattered by a non-planar surface.
In particular, for a homogeneous, isotropic medium we have:

$$\nabla \times \nabla \times \mathbf{E} - \omega^2 \epsilon \mu \mathbf{E} = i\omega\mu \mathbf{J} \ , \tag{6.7}$$

which we can write as

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = -i\omega\mu \mathbf{J} - i\omega\mu \frac{1}{k^2} \nabla\nabla \cdot \mathbf{J} \ , \tag{6.8}$$

by using the vector identity

$$\nabla \times \nabla \times \mathbf{E} = -\nabla^2 \mathbf{E} + \nabla\nabla \cdot \mathbf{E} \tag{6.9}$$

and the fact that (from Maxwell equations and the continuity equation)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} = \frac{\nabla \cdot \mathbf{J}}{i\omega\epsilon} \ .$$

We can use the identity operator $\bar{\mathbf{I}}$ to write (6.8) as

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = -i\omega\mu \left[\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2} \right] \cdot \mathbf{J} \ . \tag{6.10}$$

This is a system of three scalar equations, each of which can be solved using the free space Green's function $G_0(\mathbf{r}' - \mathbf{r})$, once we interpret $-i\omega\mu \left[\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2} \right] \cdot \mathbf{J}$ as a source term. This gives the vector solution

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int G_0(\mathbf{r}' - \mathbf{r}) \left[\bar{\mathbf{I}} + \frac{\nabla'\nabla'}{k^2} \right] \cdot \mathbf{J} d\mathbf{r}' \ . \tag{6.11}$$

It can be proven (using vector identities) that equation (6.11) can also be written as

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int \mathbf{J} \cdot \left[\bar{\mathbf{I}} + \frac{\nabla'\nabla'}{k^2} \right] G_0(\mathbf{r}' - \mathbf{r}) d\mathbf{r}' \ , \tag{6.12}$$

or, more compactly,

$$\mathbf{E}(\mathbf{r}) = i\omega\mu \int \mathbf{J} \cdot \bar{\mathbf{G}}(\mathbf{r}', \mathbf{r}) \ d\mathbf{r}' \ , \tag{6.13}$$

where

$$\bar{\mathbf{G}}(\mathbf{r}', \mathbf{r}) = \left[\bar{\mathbf{I}} + \frac{\nabla'\nabla'}{k^2} \right] G_0(\mathbf{r}' - \mathbf{r}) \tag{6.14}$$

is a dyad known as the **dyadic Green's function** for the electric field in an unbounded, homogeneous medium.
Note that (6.14) has a singularity of order $\frac{1}{|\mathbf{r}'-\mathbf{r}|^3}$ when $\mathbf{r} \to \mathbf{r}'$, so due care must be taken in evaluating the integral. It is useful, in this respect, to define the free space Green's function, in terms of its Fourier transform $G_0(|bfk)$

$$\hat{G}_0(\mathbf{r}', \mathbf{r}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{K}\cdot\mathbf{r}'-\mathbf{r}G_0}(\mathbf{k}) d\mathbf{k} \ , \tag{6.15}$$

and work in the $\mathbf{k}$ domain.

A different approach is usually more suited to polarised waves in a layered medium.
Let's first consider linearly polarised TE waves (so, e.g., $E = \hat{\mathbf{y}} E_y \equiv \mathbf{E}_y$ in an isotropic, inhomogeneous medium where the permittivity and permeability vary in one direction only, e.g. $\epsilon = \epsilon(z)$ and $\mu = \mu(z)$.
The Helmholtz equation then becomes

$$\mu \nabla \times \mu^{-1} \nabla \times \mathbf{E}_y - \omega^2 \epsilon \mu \mathbf{E}_y = i\omega\mu\mathbf{J} \ , \tag{6.16}$$

Using the vector identity (6.9) as before, and noting that from Maxwell equations we have in this case

$$\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon \mathbf{E} = \frac{\partial}{\partial y} \epsilon(z) E_y = 0 \ ,$$

so $\frac{\partial E_y}{\partial y} = 0$, the Helmholtz equation reduces further to

$$\left[\frac{\partial^2}{\partial x^2} + \mu(z) \frac{\partial}{\partial z} \mu^{-1}(z) \frac{\partial}{\partial z} + \omega^2 \mu \epsilon \right] E_y = 0 \tag{6.17}$$

TE waves can be characterised in an analogous manner by the H_z component of the magnetic field, H_z, which obeys the equation

$$\left[\frac{\partial^2}{\partial x^2} + \mu(z) \frac{\partial}{\partial z} \mu^{-1}(z) \frac{\partial}{\partial z} + \omega^2 \mu \epsilon \right] \mu H_z = 0 \ , \tag{6.18}$$

where we have used the fact that, for TE waves and a medium with spatial dependency in the z-direction only, and $\frac{\partial^2 E_y}{\partial y^2} = 0$

Similarly, for TM waves we have:

$$\left[\frac{\partial^2}{\partial x^2} + \mu(z) \frac{\partial}{\partial z} \mu^{-1}(z) \frac{\partial}{\partial z} + \omega^2 \mu \epsilon \right] E_y = 0 \ , \tag{6.19}$$

and we recover the result that TE and TM waves are decoupled in a planar, 1-dimensional inhomogeneity.

The solutions to each scalar equation (6.17) and (6.18) must have a dependency in the x-direction of the form $e^{\pm k_x x}$ for all z. Considering them together as a system of equations, we can write the solution in the form

$$\begin{bmatrix} E_y \\ H_y \end{bmatrix} = \begin{bmatrix} e_y(z)e^{\pm ik_x x} \\ h_y(z)e^{\pm ik_x x} \end{bmatrix} \tag{6.20}$$

for all z. Since the medium is translationally invariant (i.e. the properties of the medium only vary in the z-direction), the solution for all z must have the same phase variation in the x-direction, and (6.17) and (6.18) become ordinary differential equations:

$$\left[\mu(z)\frac{d}{dz}\mu^{-1}(z)\frac{d}{dz} + \omega^2\mu\epsilon - k_x^2\right] e_y \quad = 0 \quad , \tag{6.21}$$

$$\left[\epsilon(z)\frac{d}{dz}\epsilon^{-1}(z)\frac{d}{dz} + \omega^2\mu\epsilon - k_x^2\right] h_y \quad = 0 \quad , \tag{6.22}$$

which corresponds to a one-dimensional problem, to be solved with appropriate boundary conditions across any interfaces present in the medium. When ϵ and μ are function of z, the solution to (6.21) can be found numerically, e.g. with a finite difference method.
If ϵ and μ are constant, the solution will be a linear superposition of $e_y(z)e^{\pm ik_z z}$, where $k_z = (\omega^2\mu\epsilon - k_x^2)^{1/2}$.
In this case, the second order ODE derived above can be converted into a first order differential equation.

To illustrate this, let's take the ODE that governs the propagation of TM waves:

$$\left[\epsilon(z)\frac{d}{dz}\epsilon^{-1}(z)\frac{d}{dz} + k_z^2\right] \phi = 0 \ , \tag{6.23}$$

and define a new function

$$\psi = \frac{1}{i\omega\epsilon}\frac{d}{dz}\phi \ . \tag{6.24}$$

Then equation (6.23)becomes

$$\frac{d}{dz}\psi = -\frac{k_z^2}{i\omega\epsilon}\phi \ . \tag{6.25}$$

Then, equations (6.24) and (6.25) can be written in matrix form as

$$\frac{d}{dz}\mathbf{V} = \bar{\mathbf{A}} \cdot \mathbf{V} \ , \tag{6.26}$$

where $\mathbf{V}^t = [\phi, \psi]$ is the vector describing the state of the system, and the matrix $\bar{\mathbf{A}}$ is given by:

$$\bar{\mathbf{A}} = \left[\begin{array}{cc} 0 & i\omega\epsilon \\ \frac{ik_z^2}{\omega\epsilon} & 0 \end{array} \right] \tag{6.27}$$

When ϵ and k_z are constant, (6.26) has a closed form solution. In this case, let

$$\mathbf{V} = e^{\lambda z}\mathbf{V_0} \ . \tag{6.28}$$

Equation (6.26) then becomes:

$$(\bar{\mathbf{A}} - \lambda\bar{\mathbf{I}}) \cdot \mathbf{V_0} = 0 \ . \tag{6.29}$$

For non trivial $\mathbf{V_0}$, the determinant $\bar{\mathbf{A}} - \lambda\bar{\mathbf{I}}$ has to be zero, giving $\lambda = \pm ik_z$. Hence

$$\mathbf{V}(z) = C_+ e^{\pm ik_z z}\mathbf{c}_+ + C_- e^{\pm -ik_z z}\mathbf{c}_- \ , \tag{6.30}$$

where $\mathbf{c}_\pm$ are eigenvectors corresponding to the eigenvalues $\pm ik_z$.
Since $\bar{\mathbf{A}}$ is not symmetric, nor Hermitian, these eigenvectors need not be orthogonal, but the eigenvectors $\mathbf{c}_\pm$ can be normalised if necessary. The solution (6.30) can also be expressed in matrix form as

$$\mathbf{V}(z) = \bar{\mathbf{c}} \cdot e^{\pm i\bar{\mathbf{K}}_z z} \cdot \mathbf{C} \ , \tag{6.31}$$

where

$$\mathbf{C}^t = [C_+ \ C_+] \ , \tag{6.32}$$

$$e^{\pm i\bar{\mathbf{K}}_z} = \left[\begin{array}{cc} e^{ik_z z} & 0 \\ 0 & e^{-ik_z z} \end{array} \right] \tag{6.33}$$

Now, using the fact that $\bar{\mathbf{c}}^{-1} \cdot \bar{\mathbf{c}} = \bar{\mathbf{I}}$, we can rewrite (6.31) as

$$\mathbf{V}(z) = \bar{\mathbf{c}} e^{\pm i\bar{\mathbf{K}}(z-z')} \cdot \bar{\mathbf{c}}^{-1} \cdot \mathbf{V}(z') \tag{6.34}$$

If we define

$$\bar{\mathbf{S}}(z, z') = \bar{\mathbf{c}} e^{\pm i\bar{\mathbf{K}}(z-z')} \cdot \bar{\mathbf{c}}^{-1} \ , \tag{6.35}$$

we can write the solution as

$$\mathbf{V}(z) = \bar{\mathbf{S}}(z, z') \cdot \mathbf{V}(z') \ . \tag{6.36}$$

The matrix $\bar{\mathbf{S}}(z, z')$ is variously known as the ***scattering matrix***,***propagator matrix***, ***transition matrix***, ...
It relates the state vectors that relate the field at two different locations z and z'.

6.1 Reflection from a layered medium

The scattering matrix can be used to solve a three-layer problem, where two planar interfaces at $z = 0$ and $z = -d$ separate three layer stacked vertically in the z-direction, with region 1 defined by $z > 0$, region 2 by $0 < z, -d$ and region 3 defined by $-d < z$.

First, the state vector in region 1 is given by:

$$\mathbf{V}_1(z) = C_{1-}e^{-ik_{1z}z}\mathbf{c}_{1-} + RC_{1-}e^{ik_{1z}z}\mathbf{c}_{1+} = \bar{\mathbf{c}}e^{\pm i\bar{\mathbf{K}}_1 z} \cdot \begin{bmatrix} R \\ 1 \end{bmatrix} C_{1-} \ . \tag{6.37}$$

Because of the definitions of ϕ and ψ, they are continuous quantities across the interfaces. Therefore,

$$\mathbf{V}_1(0) = \mathbf{V}_2(0) \ ; \ \mathbf{V}_3(-d) = \mathbf{V}_2(-d) \ .$$

The $\bar{\mathbf{S}}$ matrix can be used to find $\mathbf{V}_2(-d)$ in terms of $\mathbf{V}_2(0)$. Consequently:

$$\mathbf{V}_3(-d) = \bar{\mathbf{S}}_2(-d, 0) \cdot \mathbf{V}_1(0) = \bar{\mathbf{S}}_2(-d, 0) \cdot \bar{\mathbf{c}}_1 \cdot \begin{bmatrix} R \\ 1 \end{bmatrix} C_{1-} \ . \tag{6.38}$$

Since region 3 should only have a transmitted wave, $\mathbf{V}_3(z)$ must be of the form

$$\mathbf{V}_3(z) = TC_{1-}e^{-ik_{3z}(z+d)}\mathbf{c_{3-}} = \bar{\mathbf{c}}_3 \cdot e^{i\bar{\mathbf{K}}_3(z+d)} \cdot \begin{bmatrix} T \\ 0 \end{bmatrix} C_{1-} \ . \tag{6.39}$$

Consequently, from (6.38) and (6.39), we conclude that

$$\bar{\mathbf{c}}_3^{-1} \cdot \bar{\mathbf{S}}_2(-d, 0) \cdot \bar{\mathbf{c}}_1 \cdot \begin{bmatrix} R \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ T \end{bmatrix} \ . \tag{6.40}$$

There are two scalar equations with two scalar unknown R and T in (6.40), and the solution can be readily found.

This approach is easily extended to the solution of **wave propagation in anisotropic, layered media**. In this more general case, instead of a state vector with two components, a state vector with at least four components is required.

We shall start with Maxwell's equations for a source-free, anisotropic medium:

$$\nabla \times \mathbf{E} = i\omega\bar{\mu} \cdot \mathbf{H} \tag{6.41}$$

$$\nabla \times \mathbf{H} = -i\omega\bar{\epsilon} \cdot \mathbf{E} \tag{6.42}$$

Given the layered medium geometry used in this section, it is convenient to work with the components of the electric and magnetic field transverse to z. We shall denote them by $\mathbf{E}_s$ and $\mathbf{H}_s$, and decompose all quantities into a transverse and a longitudinal part as follows:

$$\begin{aligned} \mathbf{E} &= \mathbf{E}_s + \mathbf{E}_z \\ \mathbf{H} &= \mathbf{H}_s + \mathbf{H}_z \\ \nabla &= \nabla_s + \hat{z}\frac{\partial}{\partial z} \end{aligned}$$

and the tensors $\bar{\mu}$ and $\bar{\epsilon}$ can be partitioned as

$$\bar{\mu} = \begin{bmatrix} \bar{\mu}_s & \bar{\mu}_{sz} \\ \bar{\mu}_{zs} & \bar{\mu}_{zz} \end{bmatrix}, \bar{\epsilon} = \begin{bmatrix} \bar{\epsilon}_s & \bar{\epsilon}_{sz} \\ \bar{\epsilon}_{zs} & \bar{\epsilon}_{zz} \end{bmatrix} \tag{6.43}$$

Here, $\bar{\mu}_s$ is 2×2, $\bar{\mu}_{sz}$ is 2×1, $\bar{\mu}_{zs}$ is 1×2 and $\bar{\mu}_{zz}$ is 1×1. Similarly for $\bar{\epsilon}$. After substituting this decomposition into (6.41), and equating transverse and longitudinal components, we have:

$$\frac{\partial}{\partial z}\hat{z} \times \mathbf{E}_s = i\omega\bar{\mu}_s \cdot \mathbf{H}_s + i\omega\bar{\mu}_{sz} \cdot \mathbf{H}_z - \nabla_s \times \mathbf{E}_z \ , \tag{6.44}$$

$$\nabla_s \times \mathbf{E}_s = i\omega\bar{\mu}_{zs} \cdot \mathbf{H}_s + i\omega\mu_{zz}\mathbf{H}_z \ . \tag{6.45}$$

We also have, either by duality, or from using the decomposition into (6.42):

$$\frac{\partial}{\partial z}\hat{z} \times \mathbf{H}_s = -i\omega\bar{\epsilon}_s \cdot \mathbf{E}_s + -i\omega\bar{\epsilon}_{sz} \cdot \mathbf{E}_z - \nabla_s \times \mathbf{H}_z \ , \tag{6.46}$$

$$\nabla_s \times \mathbf{H}_s = -i\omega\bar{\epsilon}_{zs} \cdot \mathbf{E}_s - i\omega\epsilon_{zz}\mathbf{E}_z \ . \tag{6.47}$$

We can then use (6.47) and (6.45) to express $\mathbf{E}_z$ and $\mathbf{H}_z$ in terms of $\mathbf{E}_s$ and $\mathbf{H}_s$:

$$\mathbf{E}_z = -\frac{1}{i\omega}\kappa_{zz}\nabla_s \times \mathbf{H}_s - \kappa_{zz}\bar{\epsilon}_{zs} \cdot \mathbf{E}_s \tag{6.48}$$

$$\mathbf{H}_z = \frac{1}{i\omega}\nu_{zz}\nabla_s \times \mathbf{E}_s - \nu_{zz}\bar{\mu}_{zs} \cdot \mathbf{H}_s \ , \tag{6.49}$$

where $\kappa_{zz} = \epsilon_{zz}^{-1}$ and $\nu_{zz} = \mu_{zz}^{-1}$. These two equations can then be used in (6.44) to give:

$$\begin{aligned} \frac{\partial}{\partial z}\hat{z} \times \mathbf{E}_s &= i\omega\bar{\mu}_s \cdot \mathbf{H}_s + \bar{\mu}_{sz} \cdot \nu_{zz}\nabla_s \times \mathbf{E}_s - i\omega\bar{\mu}_{sz} \cdot \bar{\mu}_{zs} \cdot \nu_{zz}\mathbf{H}_s \\ &+ \frac{1}{i\omega}\nabla_s\kappa_{zz} \times \nabla_s \times \mathbf{H}_s + \nabla_s \times \kappa_{zz}\bar{\epsilon}_{zs} \cdot \mathbf{E}_s \; . \end{aligned} \tag{6.50}$$

By duality, we have:

$$\begin{aligned} \frac{\partial}{\partial z}\hat{z} \times \mathbf{H}_s &= i\omega\bar{\epsilon}_s \cdot \mathbf{E}_s + \bar{\epsilon}_{sz} \cdot \kappa_{zz}\nabla_s \times \mathbf{H}_s - i\omega\bar{\epsilon}_{sz} \cdot \bar{\epsilon}_{zs} \cdot \kappa_{zz}\mathbf{E}_s \\ &- \frac{1}{i\omega}\nabla_s \times \nu_{zz}\nabla_s \times \mathbf{E}_s + \nabla_s \times \nu_{zz}\bar{\mu}_{zs} \cdot \mathbf{H}_s \; . \end{aligned} \tag{6.51}$$

If we assume that the fields $\mathbf{E}_s$ and $\mathbf{H}_s$ have $e^{i\{\mathbf{k}_s \cdot \mathbf{r}_s}$ in the transverse direction for all z's, and if we also assume that $\bar{\epsilon}$ and $\bar{\mu}$ are a function of z only, then taking $-\hat{z}\times$ of equations (6.50) and (6.51), and carrying out the transverse derivative ∇_s gives:

$$\begin{aligned} \frac{d}{dz}\mathbf{E}_s &= \left[(-i\omega\hat{z} \times \bar{\mu}_s \cdot) + i\omega\hat{z} \times \bar{\mu}_{sz} \cdot \bar{\mu}_{zs} \cdot \nu_{zz} - \left(\frac{i\hat{z}}{\omega} \times \mathbf{k}_s \times \kappa_{zz}\mathbf{k}_s \times\right)\right]\mathbf{H}_s \\ &+ \left[(-i\hat{z} \times \bar{\mu}_{sz} \cdot \nu_{zz}\mathbf{k}_s \times) - i\hat{z} \times \mathbf{k}_s \times \kappa_{zz}\bar{\epsilon}_{zs} \cdot\right]\mathbf{E}_s \end{aligned} \tag{6.52}$$

$$\begin{aligned} \frac{d}{dz}\mathbf{H}_s &= \left[(i\omega\hat{z} \times \bar{\epsilon}_s \cdot) - i\omega\hat{z} \times \bar{\epsilon}_{sz} \cdot \bar{\epsilon}_{zs} \cdot \kappa_{zz} + \left(\frac{i\hat{z}}{\omega} \times \mathbf{k}_s \times \nu_{zz}\mathbf{k}_s \times\right)\right]\mathbf{E}_s \\ &+ \left[(-i\hat{z} \times \bar{\epsilon}_{sz} \cdot \kappa_{zz}\mathbf{k}_s \times) - i\hat{z} \times \mathbf{k}_s \times \nu_{zz}\bar{\mu}_{zs} \cdot\right]\mathbf{H}_s \end{aligned} \tag{6.53}$$

Now these two equations can be written in matrix form as a state equation:

$$\frac{d}{dz}\begin{bmatrix} \mathbf{E}_s \\ \mathbf{H}_s \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{H}}_{11} & \bar{\mathbf{H}}_{12} \\ \bar{\mathbf{H}}_{21} & \bar{\mathbf{H}}_{22} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{E}_s \\ \mathbf{H}_s \end{bmatrix} , \tag{6.54}$$

where $\bar{\mathbf{H}}_{ij}$ are 2×2 matrices. The state equation can also be written as

$$\frac{d}{dz}\mathbf{V} = \bar{\mathbf{H}} \cdot \mathbf{V} , \tag{6.55}$$

where $\mathbf{V}^t = (\mathbf{E}_s, \mathbf{H}_s)$ is a four-component vector, and $\bar{\mathbf{H}}$ is a 4×4 matrix.

The solution to (6.55) can now be sought, in a similar way as the solution to (6.26) in the one-dimensional case.
Let

$$\mathbf{V} = e^{\lambda z}\mathbf{V_0} \; . \tag{6.56}$$

Using this in (6.55) then gives:

$$(\bar{\mathbf{H}} - \lambda\bar{\mathbf{I}}) \cdot \mathbf{V_0} = 0 \ . \tag{6.57}$$

Since $\bar{\mathbf{H}}$ is a 4×4 matrix, there will be four eigenvalues and eigenvectors, because $\det(\bar{\mathbf{A}} - \lambda\bar{\mathbf{I}}) = 0$ gives a quartic for λ. Hence, the general solution of (6.55) has the form:

$$\mathbf{V}(z) = A_1\mathbf{a_1}e^{i\beta_1 z} + A_2\mathbf{a_2}e^{i\beta_2 z}A_1\mathbf{a_3}e^{i\beta_3 z} + A_4\mathbf{a_4}e^{-i\beta_4 z} \ , \tag{6.58}$$

where $\mathbf{a}_j$ are eigenvectors corresponding to the eigenvalues $i\lambda_j$.
Again, since $\bar{\mathbf{H}}$ is not symmetric, nor Hermitian, these eigenvectors need not be orthogonal. The solution (6.58) can also be expressed in matrix form as

$$\mathbf{V}(z) = \bar{\mathbf{a}} \cdot e^{i\bar{\beta}_z z} \cdot \mathbf{A} \ , \tag{6.59}$$

where $\bar{\mathbf{a}}$ is a 4×4 matrix containing the eigenvectors $\mathbf{a}_j$:

$$\bar{\mathbf{a}} = [\bar{\mathbf{a}}_1, \bar{\mathbf{a}}_2, \bar{\mathbf{a}}_3, \bar{\mathbf{a}}_4] \tag{6.60}$$

and $\mathbf{A}$ is a column vector containing the A_i's. $\bar{\beta}$ is a diagonal matrix, where the i-th diagonal elements eigenvalues correspond to the i-th eigenvalus. So $e^{i\bar{\beta}\mathbf{z}}$ is given by

$$e^{i\bar{\beta}\mathbf{z}} = \begin{bmatrix} e^{i\beta_1 z} & 0 & 0 & 0 \\ 0 & e^{i\beta_2 z} & 0 & 0 \\ 0 & 0 & e^{i\beta_3 z} & 0 \\ 0 & 0 & 0 & e^{i\beta_4 z} \end{bmatrix} \tag{6.61}$$

These eigenvalues and eigenvectors are ordered, in such a way that the first two diagonal elements correspond to an up-going wave, and the last two to downgoing waves.
As a result, equation (6.62) can be written as:

$$\begin{aligned} \mathbf{V}(z) &= \bar{\mathbf{a}} \cdot e^{i\bar{\beta}(z-z')} \cdot \bar{\mathbf{a}}^{-1}\bar{\mathbf{a}}e^{i\bar{\beta}z'} \cdot \mathbf{A} \\ &= \bar{\mathbf{S}}(z, z') \cdot \mathbf{V}(z') \ . \end{aligned} \tag{6.62}$$

where

$$\bar{\mathbf{S}}(z, z') = \bar{\mathbf{a}} \cdot e^{i\bar{\beta}(z-z')} \cdot \bar{\mathbf{a}}^{-1} \ , \tag{6.63}$$

is the scattering matrix.

7 The inverse scattering problem

We have so far only considered the *direct* scattering problem, i.e., given a wavefield u_i incident upon an inhomogeneity (this could be an interface such as an infinite surface or a finite, closed object, or an extended inhomogeneity such as a medium with varying refractive index), we have considered ways of finding the scattered field u_s, or equivalently the total field $u = u_i + u_s$.

The *inverse* scattering problem starts from the knowledge of the scattered field u_s, and asks questions about the inhomogeneities that produced it (for example their shape, or their refractive index) and about the source field.

This area of research is fairly new, because the nature of the problem gives rise to a mathematical problem which is *ill-posed*, and until about the '60's was not considered worth studying from a mathematical point of view.

Let's see what *'well-posed'* means. According to Hadamard, a problem is ***well-posed*** if

1. There exists a solution to the problem (**existence**)
2. There is at most one solution (**uniqueness**)
3. The solution depends continuously on the data (**stability**)

For a problem expressed as

$$Ax = y \ , \tag{7.1}$$

where A is an operator from a normed space X into a normed space Y, $A : X \mapsto Y$ the requirements listed above translate into the following properties of the operator A:

1. A **is surjective**. If it isn't, then equation (7.1) is not solvable for all $y \in Y$ (*non-existence*).
2. A **is injective**. If it isn't, then equation (7.1) may have more than one solution (*non-uniqueness*)
3. The **solution depends continuously on** y, i.e. $\forall$ sequences $x_n \in X$ with $Ax_n \to Kx$ as $n \to \infty$, it follows that $x_n \to x$ as $n \to \infty$. If this is not the case, then there may be cases when for $\| y' - y \| \ll 1$ we have $\| x' - x \| \gg 1$, small differences in y (e.g. small errors in the measurement or in the numerical computation give rise to large errors in the solution (*instability*).

Absence of even one of these properties is likely to pose considerable difficulties in finding the solution to a problem.

If all the above properties apply, then the inverse operator $A^{-1} : Y \mapsto X$ exists and is bounded, and

$$\| x \| \leq C \| y \| \ , \tag{7.2}$$

where $C = \| A^{-1} \|$.

If an inverse does exist for some y, but is not bounded, then there does not exist a constant C for which (7.2) holds for all $y \in A(X)$.
It is possible, though, even when A^{-1} is not bounded, but has dense range, to construct a family of bounded approximation to A^{-1}. A strategy for achieving this is the ***Tikhonov regularisation*** procedure, which provides a mean to cope with ill-posedness.

7.1 Tikhonov regularisation

Definition A regularisation strategy for $A : X \mapsto Y$ is a family of bounded linear operators $R_\alpha : Y \to X$ for $\alpha > 0$ such that

$$R_\alpha y \to A^{-1} y \text{ as } \alpha \to \infty \tag{7.3}$$

When it is not clear whether a solution to the inverse scattering problem for (7.1) exists, it is natural, as a first attempt at computing an approximate solution, to try to find an x to minimise $\| Ax - y \|$.
It is possible to demonstrate that (**Theorem**): *For every $y \in Y$, then $x' \in X$ satisfies*

$$\| Ax' - y \| \leq \| Ax - y \| \quad .$$

if and only if *x' solves the normal equation*

$$A^* A x' = A^* y \ , \tag{7.4}$$

where $A^* : Y \mapsto X$.

Equation (7.4) is still ill-posed, if the original scattering problem was ill-posed, but this ill-posedness can be removed by introducing a small perturbation, so replacing the original problem with the slightly perturbed one below:

$$\alpha x_\alpha + A^* A x_\alpha = A^* y \tag{7.5}$$

for some small $\alpha > 0$.

7.1 Tikhonov regularisation

It is possible to prove that (**Theorem**):
If $\alpha > 0$, then the operator $(\alpha I + A^*A) : X \mapsto X$ has an inverse, which is bounded, with $\| (\alpha I + A^*A)^{-1} \| \leq \alpha^{-1}$.
Given a linear bounded operator $A : X \mapsto Y$, and $y \in Y$, the ***Tikhonov functional*** is defined by

$$J_\alpha = \| Ax - y \|^2 + \alpha \| x \|^2 \quad \forall x \in X \tag{7.6}$$

For $\alpha > 0$, the Tikhonov functional J_α, as defined above, has a unique minimum x_α given as the unique solution of the equation

$$\alpha x_\alpha + A^*Ax_\alpha = A^*y \ . \tag{7.7}$$

The solution of this equation can be written as $x_\alpha = R_\alpha y$, with

$$R_\alpha = (\alpha I + A^*A)^{-1}A^* \ : Y \mapsto X. \tag{7.8}$$

$x_\alpha = R_\alpha y$ is referred to as the *Tikhonov regularisation solution* of (7.1).
This strategy then approximates the actual solution $x = A^{-1}y$ by the regularised solution x_α, given y. In general, a y_δ will be known, which differs from y by some error δ (for example because it is experimental data):

$$\| y_\delta - y \| \leq \delta \ . \tag{7.9}$$

It is useful to be able to approximate the error involved in the regularisation, and to relate it to the error associated with incorrect initial data δ. Let's write

$$x_{\alpha(\delta)} - x = R_\alpha y_\delta - R_\alpha y + R_\alpha Ax - x \ . \tag{7.10}$$

Then, by the triangle inequality we have the estimate

$$\| x_{\alpha(\delta)} - x \| \leq \delta \| R_\alpha \| + \| R_\alpha Ax - x \| \tag{7.11}$$

This decomposition shows that the error consists of two parts: the first term reflects the influence of the incorrect data, and the second term is due to the approximation error between R_α and A^{-1}.
The regularisation scheme requires a strategy for choosing the parameter α on the basis of the error δ in the data, in order to achieve an acceptable total error for the regularised solution.

8 Methods for solving the inverse scattering problem

Inverse problems have a variety of very important practical applications, ranging from the detection of land mines, to medical imaging, analysis of subsurface strata for oil and gas recovery, reconstruction and detection of craft, missiles and submarines, non-destructive testing of materials and structures, and many more.
As mentioned previously, there are several types of inverse scattering problems. We shall concentrate first on the problem of reconstructing the geometry of the scatterer, then we shall consider the problem of reconstructing the refractive index. We shall only present a few simple results. For a comprehensive review see Sleeman (1982) IMA J. Appl. Math.**29** 113-142.

Inverse problems have been treated from many points of view.

- Some exact solutions, depending on the geometry of the scatterer, are available. They are usually based on expressing the surface of the scatterer parametrically in a coordinate system in which the Helmholtz equation is separable.
- Some methods exploit the properties of the far field in order to construct an analytical continuation of the far field into the near field of the scatterer, and the circle of minimum radius enclosing the scatterer, then determine enough points on the scatterer to approximate its shape sufficiently. The *method of Imbriale and Mittra* comes in this category.
- A number of methods based on iterative procedures are also available. These are particularly suited to lower frequencies, and for problems of scattering by extended inhomogeneities.

8.1 The method of Imbriale and Mittra

(Ref: Imbriale & Mittra 1970, IEEE Trans. Antennas & Propag. **18**, 633)
In order to describe this method, we shall first consider the properties required of a function $f(\mathbf{n}, \mathbf{k})$ to be admissible as a far field. Recall that, in the direct scattering problem, given an incident (time-harmonic) field ϕ_i on a (bounded) scatterer with boundary ∂V, we seek the total field

$$\phi = \phi_i + \phi_s \ ,$$

such that ϕ_s obeys the Helmholtz equation with suitable boundary conditions on ∂V, and the radiation condition at infinity, which can be expressed as

$$r^{1/2}\left(\frac{\partial \phi_s}{\partial r} - i|\mathbf{k}|\phi_s\right) \to 0 \quad \text{as } r \to \infty \quad \text{(in 2D)} \tag{8.1}$$

$$r\left(\frac{\partial \phi_s}{\partial r} - i|\mathbf{k}|\phi_s\right) \to 0 \quad \text{as } r \to \infty \quad \text{(in 3D)} \tag{8.2}$$

Then there exists a function $f(\mathbf{n}, \mathbf{k})$ such that

$$\phi_s(\mathbf{x}) = \frac{e^{|\mathbf{k}||\mathbf{x}|}}{|\mathbf{x}|^{1/2}}\left(f(\mathbf{n}, \mathbf{k}) + O\left(\frac{1}{|\mathbf{x}|^{1/2}}\right)\right) \quad \text{(in 2D)} \tag{8.3}$$

$$\phi_s(\mathbf{x}) = \frac{e^{|\mathbf{k}||\mathbf{x}|}}{|\mathbf{x}|}\left(f(\mathbf{n}, \mathbf{k}) + O\left(\frac{1}{|\mathbf{x}|}\right)\right) \quad \text{(in 3D)} \tag{8.4}$$

as $|\mathbf{x}| \to \infty$, where $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$. f is called the ***far field amplitude***, or also ***directivity pattern***.

A theorem due to Müller (1955) characterises the functions admissible as far field amplitudes.
Theorem: *A necessary and sufficient condition for a function $f(\mathbf{n}, \mathbf{k})$ defined on the unit sphere S^{n-1} to be a far field amplitude is that there exists a harmonic function $H(\mathbf{n}, \mathbf{k})$, analytic for all $\mathbf{x} \in \mathbb{R}^n$ and is such that $H(\mathbf{n}, \mathbf{k}) = f(\mathbf{n}, \mathbf{k})$ on S^{n-1}, and further has the property:*

$$\int_{|\mathbf{x}|=R} |H(\mathbf{n}, \mathbf{k})|^2 ds = O\left(e^{2|\mathbf{k}|CR}\right) \ , \tag{8.5}$$

where C is a non-negative constant.
When this condition is satisfied, there exists a **unique** *function $\Phi(\mathbf{n}, \mathbf{k})$ which satisfies the Sommerfeld radiation condition and is a regular solution of the Helmholtz equation for $|\mathbf{x}| > C$, such that*

$$\Phi(\mathbf{n}, \mathbf{k}) = \frac{e^{|\mathbf{k}||\mathbf{x}|}}{|\mathbf{x}|^{1/2}}\left(f(\mathbf{n}, \mathbf{k}) + O\left(\frac{1}{|\mathbf{x}|^{1/2}}\right)\right) \quad \text{(in 2D)} \tag{8.6}$$

$$\Phi_s(\mathbf{x}) = \frac{e^{|\mathbf{k}||\mathbf{x}|}}{|\mathbf{x}|}\left(f(\mathbf{n}, \mathbf{k}) + O\left(\frac{1}{|\mathbf{x}|}\right)\right) \quad \text{(in 3D)} \tag{8.7}$$

as $|\mathbf{x}| \to \infty$.

The constant C in (8.5) gives the radius of the sphere outside which $\Phi(\mathbf{n}, \mathbf{k})$ is defined. In other words, the sources generating the given far field are located within a sphere of radius C.

From the uniqueness of $\Phi(\mathbf{n},\mathbf{k})$ and $\phi_s(\mathbf{n},\mathbf{k})$, it follows that

$$\phi_s(\mathbf{n},\mathbf{k}) = \Phi(\mathbf{n},\mathbf{k}) \text{ for } |\mathbf{x}| > C \ . \tag{8.8}$$

Thus, an important problem to be considered is that of locating the region containing the sources that generate $\Phi(\mathbf{n},\mathbf{k})$. We shall therefore seek to construct an analytic continuation of $\Phi(\mathbf{n},\mathbf{k})$ into the region $|\mathbf{x}| \leq C$.

If we expand $f(\mathbf{n},\mathbf{k})$ in terms of spherical harmonics:

$$f(\mathbf{n},\mathbf{k}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{nm} Y_n^m(\mathbf{n}) \tag{8.9}$$

then, from (8.6) and the fact that $\Phi(\mathbf{n},\mathbf{k})$ is a solution of the Helmholtz equation in the exterior of V, we can write:

$$\Phi(\mathbf{n},\mathbf{k}) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} i^n h_n^{(1)}\left(|\mathbf{k}||\mathbf{x}|\right) Y_n^m(\mathbf{n}) \tag{8.10}$$

for $|\mathbf{x}| > C$, where $h_n^{(1)}\left(|\mathbf{k}||\mathbf{x}|\right)$ is either a Hankel function (in 2D) (usually denoted by $H_n^{(1)}$), or a spherical Hankel function (in 3D).
Now we need to find the analytic continuation of the series (8.10) for $|\mathbf{x}| \leq C$, and the location of the sources generating Φ.

We shall illustrate this in the 2D case with soft (Dirichlet) boundary conditions (see Crighton *et al* 1992, Ch. 19). Let us first note that, for $r \to \infty$, the asymptotic behaviour of the Hankel function is

$$H_n^{(1)}(kr) \sim \left(\frac{2}{\pi kr}\right)^{1/2} e^{ikr - in\pi/2 - 1\pi/4} \ , \tag{8.11}$$

where $kr = |\mathbf{k}||\mathbf{x}|$.
In 2D, the far field f is a function of $\theta - \theta_0$, where θ_0 is the direction of the incoming plane wave given by

$$\phi_i = e^{ikr}\cos(\theta - \theta_0) \ .$$

Therefore, from (8.11) and (8.10), we get:

$$\phi_s \sim \left(\frac{1}{kr}\right)^{1/2} e^{ikr} f(\theta) \qquad \text{as } kr \to \infty \ , \tag{8.12}$$

where

$$f(\theta) = \left(\frac{2}{\pi}\right)^{1/2} e^{-i\pi/4} \sum_{-\infty}^{\infty} A_n e^{in\theta} \ . \tag{8.13}$$

If we know $f(\theta)$ at a given wavenumber k, then the Fourier coefficients A_n are known in principle and can be computed. Then the scattered field is given by (8.10) for $|\mathbf{x}| > C$ (where C is as yet unknown). For $|\mathbf{x}| \leq C$ the series (8.10) will in general diverge, and it has to be continued analytically.

Suppose that ∂V is convex. Then, from the assumption of a soft boundary, we have that $\phi = \phi_i + \phi_s = 0$ on some convex boundary to be found.
The far field amplitude $f(\theta)$ is given, with respect to some origin $\mathbf{O}$.
Without loss of generality, we may take the coordinate system such that the incident wave propagates parallel to the x-axis, so

$$\phi_i = e^{ikx}$$

Given $f(\theta)$, we can compute in practice only a *finite* number of Fourier coefficients A_n, hence we shall have

$$f(\theta) = \left(\frac{2}{\pi}\right)^{1/2} e^{-i\pi/4} \sum_{-N}^{N} A_n e^{in\theta} \ , \tag{8.14}$$

where N is large enough to ensure a good approximation.
From (8.10) and $\phi = \phi_i + \phi_s$, the total field for $|\mathbf{x}| > C$ can be written as

$$\phi = e^{ikr\cos\theta} + \sum_{-N}^{N} i^n A_n H_n^{(1)}(kr) e^{in\theta} \ , \tag{8.15}$$

where the coefficients A_n are now known.
If this expression is computed for a succession of decreasing values of r, there will be eventually be a point on a circle of radius C_0 at which $\phi = 0$, this being one point of the boundary of the scatterer ∂V.

At this stage we know that V is inside the circle $\mathcal{L}_0(r = C_0)$, and we know

one point where ∂V touches $\mathcal{L}_0$.
It is not possible to take smaller values of r to find other points of ∂V, since the sum (8.15) will generally diverge inside $\mathcal{L}_0$. Other points of ∂V are now found by shifting the origin to a different position $\mathbf{O}_1$, given by $(r, \theta) = (r_1, \theta_1)$, say.

Let (r', θ') denote new coordinates with respect to the new origin. Since $r' \sim r - r_1$ and $\theta' \sim \theta$ as $r \to \infty$, the new far field, relative to the origin $\mathbf{O}_1$, is

$$\begin{aligned} f'(\theta) &= f(\theta) e^{ikr_1 \cos(\theta - \theta_1)} \\ &= \left(\frac{2}{\pi}\right)^{1/2} e^{-i\pi/4} \sum_{-\infty}^{\infty} A'_m e^{im\theta} \ . \end{aligned} \tag{8.16}$$

The A'_m are related to the A_n by the formula

$$\begin{aligned} \sum_{-\infty}^{\infty} A'_m e^{im\theta} &= \sum_{-\infty}^{\infty} A_n e^{in\theta} e^{ikr_1 \cos(\theta - \theta_1)} \\ &= \sum_{-\infty}^{\infty} A_n \sum_{-\infty}^{\infty} J_p(kr_1) i^p e^{ip(\theta - \theta_1)} \ , \end{aligned}$$

where $J_p(kr_1)$ are Bessel functions of the first kind. Hence

$$A'_m = \sum_{-\infty}^{\infty} i^{m-n} A_n J_{m-n}(kr_1) e^{-i(m-n)\theta_1} \ , \tag{8.17}$$

which can be computed from the $2N + 1$ known values of A_n.
Thus a large finite number of the A'_m are known, and we have a different representation

$$\phi_s = \sum_{-M}^{M} i^m A'_m H_m^{(1)}(kr') e^{im\theta'} \ , \ r' > C_1 \ , \tag{8.18}$$

which will converge outside a different circle $\mathcal{L}_1 (r' = C_1)$, centred at $\mathbf{O}_1$, where $\mathcal{L}_1$ contains V. Again, by computing $\phi = \phi_i + \phi_s$ for successive smaller values of r' until ϕ vanishes at some point with $r' = C_1$, we determine $\mathcal{L}_1$ and another point of contact with ∂V.

The process can be continued successively to limit further the region occupied by V, and to find further points on ∂V.

In practice, the far field amplitude $f(\theta)$ may not be known for all θ, but just for a limited range $\theta_1 < \theta < \theta_2$. To deal with this, Imbriale & Mittra (1970) suggest that $f(\theta)$ may be approximated by a sum

$$P_N(\theta) = \sum_{-N}^{N} \bar{A}_n e^{in\theta} \ ,$$

with coefficients $\bar{A}_n$ chosen to minimise the error indicator

$$\int_{\theta_1}^{\theta_2} |f(\theta) - P_N(\theta)|^2 d\theta \ . \tag{8.19}$$

Their results are good for the example of a circle if $\theta_1 = -135^0$ and $\theta_2 = +135^0$, and are reasonable even for $\theta_1 = -60^0$ and $\theta_2 = +60^0$.

It is clear that the process described above will not determine ∂V if there are several scatterers or if ∂V is not convex. For example, if ∂V consists of two circles, this procedure would provide only the information that ∂V lies within the region bounded by the two common tangents to the circles, and the two 'outer' arcs from the tangent points:

Imbriale & Mittra provide a straightforward extension to deal with such cases. This is based on the fact that any solution of the Helmholtz equation that is regular at and near some origin $\mathbf{O}_1$ can be expanded in the form:

$$\phi_s = \sum_{-\infty}^{\infty} B_n J_n(kr') e^{in\theta'} \ , \ r' > D_1 \ , \tag{8.20}$$

where D_1 is the distance from $\mathbf{O}_1$ to the nearest singularity. Here this ensures that (8.20) converges *within* a circle $\mathcal{K}_1(r' = D_1)$ that just touches ∂V.

The difference between the exterior expansion (8.10) and the interior expansion (8.20) is that $H_n^{(1)}(kr)$ satisfies the Sommerfeld radiation condition, but is singular at $r = r'$; whilst $J_n(kr')$ is regular at $r = r'$, but does not correspond to outward-travelling waves at infinity.

Thus (8.20) is valid within the circle $\mathcal{K}_1$. The coefficients B_n can be obtained in terms of A_n, provided the inside of $\mathcal{K}_1$ and the outside of $\updownarrow_0$ overlap. This can be ensured by choosing $\mathbf{O}_1$ outside $\updownarrow_0$.
Then comparing the two expansions (8.10) and (8.20), and using the following addition theorem for Bessel functions:

$$H_n^{(1)}(kr)e^{in(\theta-\theta_1)} = \sum_{m=-\infty}^{\infty} H_{n-m}^{(1)}(kr_1)J_m(kr')e^{im(\theta'-\theta)} \ ,$$

one finds:

$$B_m = \sum_{n=-\infty}^{\infty} i^n A_n H_{n-m}^{(1)}(kr_1)e^{i(n-m)\theta_1} \ , \qquad (8.21)$$

Thus the B_m are known in principle, or rather, in practice, a large finite number of them are known, and the representation (8.20) can be considered as being known.
Now we take increasing values of r', until ϕ vanishes at some point $r' = D_1$ and $\theta' = \theta_1$. This ensure that ∂V lies *outside* the circle $\mathcal{K}_1(r' = D_1)$ and gives a point of contact with $\mathcal{K}_1$.
In a similar way to that used previously for the circles $\mathcal{L}_n$, we can extend this process by choosing another origin $\mathbf{O}_2$, given by $(r', \theta') = (r_2, \theta_2)$, choosing $\mathbf{O}_2$ to lie inside $\mathcal{K}_1$. If (r'', θ'') denote polar coordinates with respect to $\mathbf{O}_2$, then we seek to continue the limited representation (8.20) by re-expanding about the new origin $\mathbf{O}_2$:

$$\phi_s = \sum_{-\infty}^{\infty} B_n'' J_n(kr'')e^{in\theta''} \ , \ r'' > D_2 \ , \qquad (8.22)$$

for some value D_2 (as yet unknown).
A comparison of (8.22) and (8.20), both valid in some overlap region, and the addition theorem for Bessel functions

$$H_n(kr')e^{in(\theta'-\theta_2)} = \sum_{m=-\infty}^{\infty} J_{n-m}(kr_2)J_m(kr'')e^{im(\theta''-\theta_2)} \ ,$$

leads to the result

$$B_m'' = \sum_{n=-\infty}^{\infty} i^n B_n J_{n-m}(kr_2)e^{-i(n-m)\theta_2} \ , \qquad (8.23)$$

and the representation (8.22) can be taken as known.
Taking successively bigger values of r'' until $\phi_i + \phi_s$ vanishes at some point on a circle $\mathcal{K}_2$, we obtain another point of ∂V, and also the information that ∂V is outside $\mathcal{K}_2$. This process can be repeated indefinitely, in principle, to get a series of circles $\mathcal{K}_1, \mathcal{K}_2, ...$ outside which ∂V must lie.

This technique presents several considerable numerical problems, since it involves many sums that have to be truncated. Also, for example, it turns out that the coefficient B_n in (8.21) is sensitive to the values of A_m at large m. These problems are discussed by Imbriale and Mittra (1970), who give results for a pair of circles of radii a and separation $2b$ between their centres, with $ka = 1$ and $kb = 2.5$.

The methods of Imbriale and Mittra seem to give reasonable reconstructions for modest values of kd (where d is the characteristic dimension of the scatterer). Nevertheless, such procedures based on analytical continuations are inherently ill-posed and subject to numerical instability. Other methods are more stable.

8.2 Optimization method

This method was described by Colton & Monk (1987), for the scattering from an acoustically soft surface S, recasting the problem as one in optimazation. It overcomes the numerical difficulties associated with the analytical continuation procedure.

Suppose that a time-harmonic plane wave with velocity potential given by

$$\phi_i = \text{Re}\left[e^{ik\mathbf{w}\cdot\mathbf{x}}\right] \tag{8.24}$$

is incident upon a scattering surface S taht encloses the origin. The total potential $\phi = \phi_i + \phi_s$ satisfies the Helmholtz equation and the boundary condition

$$\phi\mathbf{x} = 0 \ , \quad \text{for } \mathbf{x} \text{ on } S \ . \tag{8.25}$$

It is assumed that k^2 is not one of the interior eigenvalues of the interior Dirichlet problem. and that the scatterer is "starlike", i.e. its surface can be represented in the form

$$\mathbf{x} = r_s(\mathbf{e})\mathbf{e} \ ,$$

where $\mathbf{e} = \mathbf{x}/(|\mathbf{x}|)$, and r_s is single-valued.

Given the far field amplitude $f(\mathbf{w}, \mathbf{e}, k)$ (in 3D) defined by

$$\phi_s \sim \left(\frac{1}{kr}\right) e^{ikr} f(\mathbf{w}, \mathbf{e}, k) \qquad\qquad \text{as } kr \to \infty \ , \tag{8.26}$$

at fixed k, the problem is to determine the function $r_s(\mathbf{e})$ that specifies tha scattering surface S.

Colton & Monk relate the far field $f(\mathbf{w}, \mathbf{e}, k)$ to a function $\psi(\mathbf{x}, k)$ that correspond to the scattered potential inside S induced by a point at the origin, i.e the function $\psi(\mathbf{x}, k)$ which satisfies

$$(\nabla^2 + k^2)\psi(\mathbf{x}, k) = 0 \quad \mathbf{x} \text{ inside} S \ , \tag{8.27}$$

with

$$\psi(\mathbf{x}, k) = \frac{e^{ikr_s}}{4\pi r_s} \ , \quad \mathbf{x} \text{ on} S \ , \tag{8.28}$$

By using Green's function formalism applied to the potential $\phi(\mathbf{x})$, with the Green's function

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{ikr}}{4\pi r} \ , \quad \text{with } r = |\mathbf{y} - \mathbf{x}| \ ,$$

in the region outside s, we can write

$$\phi(\mathbf{y}) = \phi_i(\mathbf{y}) - \frac{1}{4\pi}\int_S \frac{e^{ikr}}{r}\frac{\partial\phi(\mathbf{x})}{\partial n} d\mathbf{x} \ , \tag{8.29}$$

where n denotes the outward normal from s. It follows that the far field amplitude f has the representation

$$f(\mathbf{w}, \mathbf{e}, k) = -\frac{k}{4\pi} \int_S e^{-ik\mathbf{e}\cdot\mathbf{x}} \frac{\partial \phi(\mathbf{x})}{\partial n} d\mathbf{x} \; . \tag{8.30}$$

Now define S_1 to be the sphere of unit radius and centre at the origin. The above identity (8.29) can be multiplied by a suitable function $g(\mathbf{e})$ and integrated with respect to $\mathbf{e}$ over the unit sphere S_1, to get

$$-\frac{4\pi}{k} \int_{S_1} f(\mathbf{w}, \mathbf{e}, k) g(\mathbf{e}) d\mathbf{e} = \int_S \psi(\mathbf{x}) \frac{\partial \phi(\mathbf{x})}{\partial n} d\mathbf{x} \; . \tag{8.31}$$

where

$$\psi(\mathbf{x}) = \int_{S_1} g(\mathbf{e}) e^{-ik\mathbf{e}\cdot\mathbf{x}} d\mathbf{e} \tag{8.32}$$

satisfies the Helmholtz equation if the kernel function $g(\mathbf{e})$ is sufficiently smooth.

Functions of the form (8.32) are called **Herglotz wave functions**, with Herglotz kernel $g(\mathbf{e})$.

It is now assumed that the domain inside S is such that the interior potential $\psi(\mathbf{x})$ defined by (8.28) and (8.30) can be represented as an Herglotz function. In this case, the integral (8.31) has value unity. This follows since, from (8.31) and (8.28), and using the boundary condition, we have:

$$\begin{aligned}
-\frac{4\pi}{k} \quad & \textstyle\int_{S_1} \quad f(\mathbf{w}, \mathbf{e}, k) g(\mathbf{e}) d\mathbf{e} = \frac{1}{4\pi} \int_S \frac{e^{ikr}}{r} \frac{\partial \phi(\mathbf{x})}{\partial n} d\mathbf{x} \\
= \frac{1}{4\pi} \quad & \textstyle\int_S \quad \left[\frac{\partial \phi}{\partial n} \frac{e^{ikr}}{r} - \phi \frac{\partial}{\partial n} \left(\frac{e^{ikr}}{r} \right) \right] d\mathbf{x} \\
= \frac{1}{4\pi} \quad & \textstyle\int_S \quad \left[\frac{\partial \phi_i}{\partial n} \frac{e^{ikr}}{r} - \phi_i \frac{\partial}{\partial n} \left(\frac{e^{ikr}}{r} \right) \right] d\mathbf{x}
\end{aligned} \tag{8.33}$$

The last step follows from the fact that the integral

$$I = \int_{S'} \left[G \frac{\partial \phi_s}{\partial n} \right] d\mathbf{x} \; , \tag{8.34}$$

is invariant with respect to any surface S' on or outside S, by virtue of Green's formula applied to ϕ_s and G. Taking S' to be a sphere of large radius R_0, one finds, by using the radiation condition satisfied by ϕ_s, that $I \to 0$ as $R_0 \to \infty$, hence $I \equiv 0$. Finally, the integral (8.33) is seen to have the value

$\phi_i(0) = 1$, from Green's formula applied to ϕ_i and G, with $(\nabla^2 + k^2)G = \delta\mathbf{x}$.
Thus

$$-\frac{4\pi}{k}\int_{S_1} f(\mathbf{w}, \mathbf{e}, k)g(\mathbf{e})d\mathbf{e} = 1 \qquad (8.35)$$

for all directions of incidence $\mathbf{w}$.
The problem is now specified by the two identities (8.35) and (8.28), to determine $g(\mathbf{e})$, then r_s.
Colton & Monk (1987) formulate the optimization problem to minimise

$$\sum_{n=1}^{N}\left|\int_{S_1} \frac{4\pi}{k}\ f(\mathbf{w}_n, \mathbf{e}, k)g(\mathbf{e})d\mathbf{e} + 1\right|^2 \qquad (8.36)$$

with respect to $g(\mathbf{e})$ from a suitable function class. Given g, hence ψ from equation (8.32), there is a second optimization problem to minimize

$$\int_{S_1}\left|\psi(r_s(\mathbf{e}) - \frac{e^{ikr_s}}{4\pi r_s}\right|^2 d\mathbf{e} \qquad (8.37)$$

with respect to $r_s(\mathbf{e})$ from a suitable function class.
The estimate for r_s gives an approximation to the surface S.
Colton & Monk (1987) give results for several axially symmetric problems, using trial functions in the form of Fourier series in the azimuthal angle. Their results give excellent reconstructions for a variety of shapes, such as the oblate spheroid, the "peanut" shape, and the "acorn" shape.

8.3 Inverse scattering in the Born approximation

When the scattering is sufficiently weak, the inverse scattering problem can be linearised and solved using the first Born (or Rytov) approximation (see Chapter 3, were these approximationas are introduced for the direct scattering problem). In this case, the (known) scattered field is written as the first Born (or Rytov) solution of the direct scattering problem, then the Fourier transform of the scattered field is related to the Fourier transform of the 'scattering potential' of the object, or medium, thus formally solving the inverse problem.

We shall consider first the Born approximation. We recall (see Chapter 3) that, given some inhomogeneity with refractive index $n(\mathbf{r})$, and a non-scattering background with refractive index 1, the the total field satisfies

$$\nabla^2\psi + k^2(\mathbf{r})\psi = 0 \ , \tag{8.38}$$

where

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0(1 + n_\delta(\mathbf{r})) \ , \tag{8.39}$$

We shall assume $n_\delta(\mathbf{r}) \ll 1$. Substituting $k_0 n(\mathbf{r})$ into (**??**) we get:

$$\nabla^2\psi + k_0^2(\mathbf{r})\psi = -k_0^2(n^2(\mathbf{r}) - 1)\psi \equiv -V(\mathbf{r})\psi \ , \tag{8.40}$$

and the scattered field is then given by

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ . \tag{8.41}$$

The total field is then given by

$$\psi = \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}')[V(\mathbf{r}')\psi(\mathbf{r}')]d\mathbf{r}' \ , \tag{8.42}$$

and the scattered field can be approximated first Born approximation by

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}')[V(\mathbf{r}')\psi_i(\mathbf{r}')]d\mathbf{r}' \ , \tag{8.43}$$

Here $G(\mathbf{r} - \mathbf{r}')$ is the free space Green's function in 3 dimension, i.e.

$$G(\mathbf{r} - \mathbf{r}') = \frac{e^{ik_0 r}}{r} \ . \tag{8.44}$$

We shall now use the following representation for

$$\frac{e^{ik_0 r}}{r} \ :$$

Banos 1966, Wolf 1969):

$$G(\mathbf{r}-\mathbf{r}') = \frac{ik_0}{2\pi}\int\int_{-\infty}^{\infty}\frac{1}{m}e^{ik_0[p(x-x')+q(y-y')+m(z-z')]}dpdq\ , \tag{8.45}$$

where:

$$m = (1-p^2-q^2)^{1/2} \text{ when } (p^2+q^2)<1 \tag{8.46}$$
$$m = i(p^2+q^2-1)^{1/2} \text{ when } (p^2+q^2)>1 \tag{8.47}$$

If we now substitute this expression for the Green's function into equation (8.43), we obtain:

$$\psi_s(\mathbf{r}) = \int\int_{-\infty}^{\infty} A^{(\pm)}(p,q;p_0,q_0)e^{ik_0(px+qy\pm mz)}dpdq\ , \tag{8.48}$$

where

$$A^{(\pm)}(p,q;p_0,q_0) = -\frac{ik_0}{8\pi^2 m}\int V\mathbf{r}'e^{ik_0[(p-p_0)x'+(q-q_0)\pm(m-m_0)z']}d\mathbf{r}'\ . \tag{8.49}$$

Here, the upper sign(+) applies in the region $\mathcal{R}^+$ where $z-z'>0$, and the lower one (-) in the region $\mathcal{R}^-$ where $z-z'<0$. Equation (8.48) represents the scattered field as an angular spectrum of plane waves, and the spectral amplitude function $A^{(\pm)}(p,q;p_0,q_0)$ is expressed in term of the scattering potential by (8.49). For homogeneous waves, i.e. when m is real, we obtain the relation:

$$A^{(\pm)}(p,q;p_0,q_0) = -\frac{ik_0}{8\pi^2 m}\hat{F}[k_0[(p-p_0),k_0(q-q_0),k_0\pm(m-m_0)]\ , \tag{8.50}$$

where $\hat{F}$ is the Fourier inverse of F:

$$\hat{F}(u,v,w) = \frac{1}{(2\pi)^3}\int F(x,y,z,)e^{ik_0(ux+vy+wz)}dxdydz\ . \tag{8.51}$$

Consider now the scattered field ψ_s in two fixed planes $z=z^+$ and $z=z^+$, situated respectively in $\mathcal{R}^+$ and $\mathcal{R}^-$.

Now, by taking the inverse Fourier transform of (8.48), with z at the fixed values z^+ and z^+, we obtain

$$A^{(\pm)}(p,q;p_0,q_0) = k_0^2 e^{\mp ik_0 mz^{\pm}}\hat{\psi}_s(k_0p,k_0q,z^{\pm})\ , \tag{8.52}$$

where

$$\hat{\psi}_s(u,v,z^{\pm}) = \frac{1}{(2\pi)^2}\int\int_{-\infty}^{\infty}e^{-i(ux+vy)}dxdy \tag{8.53}$$

is the inverse Fourier transform of ψ_s with respect to the variables x and y. Now, comparing (8.52) and (8.52), and using $m = (1 - p^2 - q^2)^{1/2}$, we obtain

$$\hat{V}(u', v', w'^{\pm}) = \frac{iw}{\pi} e^{\mp iwz^{\pm}} \hat{\psi}_s(u, v, w^{\pm}) \ , \tag{8.54}$$

where

$$\begin{aligned} u' &= u - k_0 p_0 \\ v' &= v - k_0 q_0 \\ w' &= \pm w - k_0 m_0 \end{aligned} \tag{8.55}$$

and

$$w = (k_0^2 - u^2 - v^2)^{1/2} \ . \tag{8.56}$$

Equation (8.54) shows that some of the three-dimensional Fourier components of the scattering potential v, and therefore the unknown refractive index, can be immediately determined by the two-dimensional components of the scattered field in the two planes $z = z^+$ and $z = z^-$.

Note that (8.54) is valid only for those two-dimensional Fourier components of $\hat{\psi}_s$ and ψ_s about which the information is carried by homogeneous waves, i.e. those for which $u^2 + v^2 \leq k_0^2$. In general, it is impossible to reconstruct inverse data associated with the high spectral components for which the information is carried by evanescent waves, because these waves decay very rapidly from the scatterer and do not contribute to the far field. This limitation arises because the problem is ill-posed.
We saw earlier that one way to obviate the limitations caused by ill-posedness is to use the Tikhonov (or other) regularization. In this case then, if we represent by $\mathbf{A}$ the integral operator in (8.43):

$$\mathbf{A}V(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r'}) V(\mathbf{r'}) \psi_i(\mathbf{r'}) d\mathbf{r'} \ , \tag{8.57}$$

then the problem we need to solve is

$$\mathbf{d} = \mathbf{A}V(\mathbf{r}) \tag{8.58}$$

where $\mathbf{d}$ is the vector of the scattered field measurements. This can be regularised by minimising the Tikhonov functional

$$J_\alpha = \parallel \mathbf{A}V(\mathbf{r}) - \mathbf{d} \parallel^2 + \alpha \parallel x \parallel^2 \tag{8.59}$$

with the penalty parameter α usually chosen based on knowledge of the noise level.

8.3 Inverse scattering in the Born approximation

Using the first Born approximation for the inverse scattering problem reduces the non-linear inverse problem to a completely linear one.
We can retain some non-linearity either by adding higher order terms in the Born approximation, or by using the ***distorted-wave Born approximation*** (DWBA). In the DWBA, instead of approximating the 'zero order' solution with the incident field as in the first Born illustrated above, we start with a perturbed field, in other words, instead of writing the refraction index as

$$n(\mathbf{r}) = 1 + n_\delta \ , \tag{8.60}$$

we write

$$n^2(\mathbf{r}) = n_0^2(\mathbf{r}) + \epsilon n_1 + \epsilon^2 n_2 + \ldots \tag{8.61}$$

The DWBA then is obtained by seeking a solution of the Helmholtz equation (8.38) in the form:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \epsilon\psi_1(\mathbf{r}) + \ldots \tag{8.62}$$

The solution terms in this series can be computed by solving:

$$\begin{aligned}
(\nabla^2 + k_0^2 n_0^2 \mathbf{r})\psi_0 &= 0 \\
(\nabla^2 + k_0^2 n_0^2 \mathbf{r})\psi_1 &= -k_0^2 n_1 \psi_0 \\
(\nabla^2 + k_0^2 n_0^2 \mathbf{r})\psi_2 &= -k_0^2 n_2 \psi_0 - k_0^2 n_1 \psi_1 \\
&\ldots
\end{aligned} \tag{8.63}$$

So the integral equation corresponding to (8.43) is now:

$$\psi_s(\mathbf{r}) = \int G^{(k)}(\mathbf{r}-\mathbf{r}')[V(\mathbf{r}')\psi_i(\mathbf{r}')]d\mathbf{r}' \ , \tag{8.64}$$

and $G^{(k)}(\mathbf{r}-\mathbf{r}')$ is not the free space Green's function any more. If $n_0^2(\mathbf{r}) = 1$, then the DWBA coincides with the Born approximation. In the DWBA it is also possible of course to go to higher terms and include more iterations.
It should be noted, though, that in general, if the measured data is contaminated with noise, so that the actual total field ψ^a is:

$$\psi^a(\mathbf{r}) = \psi(\mathbf{r}) + \Delta(\mathbf{r}) \ , \tag{8.65}$$

where $\Delta(\mathbf{r})$ is the noise, then

$$\psi_s(\mathbf{r}) = \psi(\mathbf{r}) - \psi_i(\mathbf{r}) + \Delta(\mathbf{r}) \ . \tag{8.66}$$

Hence, as successive iterations improve on $\psi_i(\mathbf{r})$ so that it is closer to $\psi(\mathbf{r})$, $\psi_s(\mathbf{r})$ is swamped by noise. Other variants of the Born iterative method are more robust and also less time-consuming, especially in higher dimensions.

9 References and further reading

References

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Chapters 4 and 5 are in good part based on earlier lecture notes by Dr Mark Spivack.

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A good book for complex methods, and specifically also topics of relevance to wave scattering, such as asymptotic approximations and the Wiener-Hopf method, is:
M.J. Ablowitz and A.S. Fokas *Complex variables*, CUP, 1997

More on asymptotic Methods: Hinch *Perturbation Methods*, Cambridge: CUP 1991

The following book gives a very useful compendium of the main known solutions to 'canonical' scattering problems (including the asymptotic solutions) for both acoustic and electromagnetic waves:
J.J. Bowman, T.B.A. Senior and P.L.E Uslenghi *Electromagnetic and acoustic scattering by simple shapes*, North-Holland, 1969