Example Sheet 2

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Starred questions are not necessarily harder, just less central to subsequent course material. Send comments and queries to `J.H.P.Dawes@damtp.cam.ac.uk`.

1. Identify the bifurcations in the following systems and sketch the bifurcation diagrams. Construct co-ordinate changes (first moving the bifurcation point to the origin) in order to put the equations explicitly in normal form.

   (a) $\dot{x} = \mu - 2x - 2x^2$,

   (b) $\dot{x} = 2\mu - (2+\mu)x + x^2$,

   (c) $\dot{x} = (\mu - 2) + \mu x + 3x^2 + x^3$,

   (d) $$\begin{aligned} \dot{x} &= (\mu+2)x + 2y - (2x^2 + 2xy + y^2)x, \\ \dot{y} &= -4x + (\mu - 2)y - (2x^2 + 2xy + y^2)y. \end{aligned}$$

   In (d), first make the linear change of co-ordinates that brings the equation into normal form, then sketch phase portraits for μ above and below the bifurcation point.

2. [Part III 2005, Paper 65, Question 2(a)]. A predator–prey interaction is described by the ODEs

$$\begin{aligned} \dot{x} &= \left(\frac{1}{a} - 1\right)x - \frac{1}{a}x^2 - xy, \\ \dot{y} &= \frac{1}{b}xy - y, \end{aligned}$$

   where a and b are positive parameters and $b \leq 1$. Find the equilibrium points and investigate their stability. Hence sketch phase portraits of the quadrant $x, y \geq 0$ in each of the three regions of the (a, b) plane that show qualitatively different dynamics.

3. Identify the bifurcation in the following system and sketch the bifurcation diagram and phase portraits:

$$\begin{aligned} \dot{x} &= \mu y - x - 2x^3, \\ \dot{y} &= x - y - y^3. \end{aligned}$$

   Compute the *extended* centre manifold near the bifurcation point to determine the nature of the bifurcation. Remember to shift the bifurcation parameter so that the bifurcation occurs when it is zero. *Hint:* if you wish you can make a linear change of co-ordinates to diagonalise the linear part of the problem first (so that $\mathbb{E}^c$ is an axis). Otherwise, just start by constructing W^c_{loc} to be tangent to $\mathbb{E}^c$ in the original co-ordinates.

4. Find the value of the (real) parameter μ at which there is a bifurcation from the trivial solution of the system

$$\begin{aligned} \dot{x} &= y - x - x^2, \\ \dot{y} &= \mu x - y - y^2. \end{aligned}$$

   Find the evolution equation on the *extended* centre manifold correct to third order. Hence deduce the nature of the bifurcation.

5. (a) The Lorenz equations

$$\begin{aligned}\dot{a} &= \sigma(-a + rb),\\ \dot{b} &= a - b - ac,\\ \dot{c} &= \varpi(-c + ab),\end{aligned}$$

clearly have an equilibrium point at the origin (the 'trivial' equilibrium). For what range of values of the parameters r, σ and ϖ (all positive) do other (non-trivial) equilibria exist? At what parameter values are there local bifurcations? Compute the centre manifold at $r = 1$ and hence determine whether the bifurcation is subcritical or supercritical.

(b) Now analyse the bifurcation at $r = 1$ using adiabatic elimination, as follows. Write $r = 1 + \mu$. Since c decays fast at $r = 1$ (so $\dot{c} \approx 0$), scale $a = \varepsilon a'$, $b = \varepsilon b'$ but $c = \varepsilon^2 c'$ to balance $c' \sim a'b'$ in the third equation. Then also substitute

$$\frac{d}{dt} = \varepsilon^\alpha \frac{d}{dt'}, \qquad \mu = \varepsilon^\beta \mu',$$

for some (as yet undetermined) positive constants α, β. Hence, re-arranging the c and b equations and substituting them into themselves we get, dropping the primes:

$$\begin{aligned}c &= ab - \varepsilon^\alpha \dot{c}/\varpi,\\ b &= a - \varepsilon^\alpha \dot{a} - \varepsilon^2 a^3 + O(\varepsilon^{\alpha+2}, \varepsilon^4).\end{aligned}$$

Substitute these into the scaled $\dot{a}$ equation and choose appropriate values for α and β to balance terms and obtain

$$\dot{a} = \frac{\sigma}{1+\sigma}(\mu a - a^3) + O(\varepsilon^2)$$

which should agree with your answer to part (a).

6. (a) For the ODEs

$$\begin{aligned}\dot{x} &= -2x + y - x^2,\\ \dot{y} &= xy - x^2,\end{aligned}$$

compute the centre manifold $x = h(y)$ to sufficiently high order to determine the stability of the equilibrium at the origin, remembering to include a linear term in $h(y)$ to ensure that W^c_{loc} is tangent to $\mathbb{E}^c$ at $(0, 0)$. Sketch $\mathbb{E}^s$, $\mathbb{E}^c$ and W^c_{loc}.

(b) For the same ODEs as in part (a), compute the first two terms of the centre manifold 'the other way around' by writing $y = \tilde{h}(x)$, again including a linear term. Notice that h and $\tilde{h}$ are inverses of each other. For this particular example we find $y = \tilde{h}(x) = 2x + \frac{3}{2}x^2$ and all higher-order terms vanish: this is the global centre manifold W^c. Check this by computing $dy/dx = \dot{y}/\dot{x}$ evaluated on W^c_{loc} and comparing this with $dy/dx = d\tilde{h}/dx$.

7. Suppose that the 2D system

$$\begin{aligned}\dot{x} &= f(x, y, \mu)\\ \dot{y} &= g(x, y, \mu)\end{aligned}$$

has an equilibrium at the origin for all μ and has a Jacobian matrix $\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$ evaluated at the origin, when $\mu = 0$. Then we would expect the system to undergo a Hopf bifurcation at $\mu = 0$. By near-identity transformations we can put the (x, y) system into the normal form

$$\begin{aligned}\dot{u} &= -\omega v + (Au + Bv)(u^2 + v^2)\\ \dot{v} &= \omega u + (Av - Bu)(u^2 + v^2)\end{aligned}$$

The coefficient A determines whether the Hopf bifurcation is subcritical or supercritical; it can be explicitly calculated in terms of partial derivatives of f and g evaluated at $x = y = \mu = 0$, which given the above assumptions leads to the expression:

$$\begin{aligned} A &= \frac{1}{16}\left(f_{xxx} + g_{xxy} + f_{xyy} + g_{yyy}\right) \\ &\quad + \frac{1}{16\omega}\Big\{ f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy} \Big\} \end{aligned}$$

(see Glendinning, p227). Consider the case where there is a conserved quantity $H(x, y)$ at $\mu = 0$, so that

$$\begin{aligned} \dot{x} &= f(x, y, 0) = \frac{\partial H}{\partial y} \\ \dot{y} &= g(x, y, 0) = -\frac{\partial H}{\partial x} \end{aligned}$$

Compute A in this case and explain what goes wrong.

8. <u>*Normal form for period-doubling bifurcation.*</u> Show that the map $x_{n+1} = f(x_n, \mu)$, where

$$f(x, \mu) = -(1 + \mu)x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots,$$

and $a_2, a_3, \ldots$ are constants, can be transformed at the period–doubling bifurcation point $\mu = 0$ into the normal form $y_{n+1} = g(y_n)$ where $g(y)$ is an odd function of y. Hint: first set $\mu = 0$, then use the near-identity transformation $y = x + \alpha x^k$ (for some $k \geq 2$) to try to eliminate the term $a_k x^k$ by choosing α. These near-identity changes of co-ordinates can be carried out repeated at successive orders.
What happens to the co-ordinate transformation procedure if μ is not zero?

* 9. <u>*Hopf bifurcation normal form transformations*</u>. Show that the set of ODEs

$$\begin{aligned} \dot{a} &= b, \\ \dot{b} &= -\lambda a + \kappa b + P a^3 + Q a^2 b, \end{aligned}$$

where P and Q are constants, and κ and λ are parameters, has a Hopf bifurcation when $\kappa = 0$ and $\lambda > 0$. Show that this bifurcation is supercritical when $Q < 0$ and subcritical when $Q > 0$. Hint: do this by transforming the ODEs into the Hopf normal form as follows. First set $\kappa = 0$ and do a linear rescaling of a, b and time to get the ODEs into the form

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \tilde{P}x^3 + \tilde{Q}x^2 y, \end{aligned}$$

where $\tilde{P}$ and $\tilde{Q}$ are constants. Then make a near-identity transformation of the form

$$\begin{aligned} x &= u + \alpha_1 u^3 + \beta_1 u^2 v + \gamma_1 u v^2 + \delta_1 v^3, \\ y &= v + \alpha_2 u^3 + \beta_2 u^2 v + \gamma_2 u v^2 + \delta_2 v^3, \end{aligned}$$

choosing α_1 etc so that the ODEs for u and v are the Hopf normal form:

$$\begin{aligned} \dot{u} &= v + (Au + Bv)(u^2 + v^2), \\ \dot{v} &= -u + (Av - Bu)(u^2 + v^2). \end{aligned}$$

* 10. <u>*Takens–Bogdanov normal form transformations*</u>. Find choices of the coefficients α_1 etc for the near–identity transformation

$$\begin{aligned} x &= \xi + \alpha_1 \xi^2 + \beta_1 \xi\eta + \gamma_1 \eta^2, \\ y &= \eta + \alpha_2 \xi^2 + \beta_2 \xi\eta + \gamma_2 \eta^2, \end{aligned}$$

which reduce the equations

$$\begin{aligned}\dot{x} &= y + a_1 x^2 + b_1 xy + c_1 y^2,\\ \dot{y} &= a_2 x^2 + b_2 xy + c_2 y^2,\end{aligned}$$

(where the coefficients $a_1, \ldots, c_2$ are given constants) to each of the simpler forms that follow:

(a) *Takens's version:*

$$\begin{aligned}\dot{\xi} &= \eta + A\xi^2 + O(3)\\ \dot{\eta} &= B\xi^2 + O(3)\end{aligned}$$

where A, B are constants, in fact linear combinations of the coefficients $a_1, \ldots, c_2$ and $O(3)$ denotes third and higher-order terms.

(b) *Bogdanov's version:*

$$\begin{aligned}\dot{\xi} &= \eta + O(3)\\ \dot{\eta} &= C\xi^2 + D\xi\eta + O(3)\end{aligned}$$

where C, D are again linear combinations of $a_1, \ldots, c_2$.

Check that (a) is transformed into (b) with coefficients $C = D = 1$ by the (nonlinear) co-ordinate transformation

$$\begin{aligned}\xi' &= \frac{4A^2}{B}\xi\\ \eta' &= \frac{8A^3}{B^2}(\eta + A\xi^2)\\ t' &= \frac{B}{2A}t, \quad \text{hence} \quad \frac{d}{dt} = \frac{B}{2A}\frac{d}{dt'}\end{aligned}$$

which involves reversing the direction of time if $AB < 0$. This demonstrates the topological equivalence of (a) and (b).

(c) *The equivariant normal form.* From the normal form symmetry theorem due to Elphick *et al* there must exist a version of the normal form which commutes with $\exp(sL^T)$ where $L = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is the linearisation of the original ODEs for (x, y), hence $\exp(sL^T) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}$ (check this). This normal form is

$$\begin{aligned}\dot{\xi} &= \eta + A\xi^2 + O(3)\\ \dot{\eta} &= A\xi\eta + B\xi^2 + O(3)\end{aligned}$$

where A, B are linear combinations of $a_1, \ldots, c_2$. Find choices of the coefficients $\alpha_1, \ldots, \gamma_2$ which produce this normal form and verify the equivariance condition

$$\mathbf{f}\left(\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\begin{pmatrix} \xi \\ \eta \end{pmatrix}\right) = \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\mathbf{f}\begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

where

$$\mathbf{f}\begin{pmatrix} \xi \\ \eta \end{pmatrix} = A\begin{pmatrix} \xi^2 \\ \xi\eta \end{pmatrix} + B\begin{pmatrix} 0 \\ \xi^2 \end{pmatrix}.$$

Note that in this example, unfortunately, the linear term L does not commute with $\exp(sL^T)$ and so the entire normal form is not $\exp(sL^T)$-equivariant.