Fokas Method (unified transform)

January 25, 2017

1 Brief History

There exists a particular class of nonlinear PDEs called integrable. By the mid eighty's the initial value problem of integrable evolution equations in one and two space variables was solved via the so called inverse scattering transform. Following this development, the outstanding open problem in the analysis of these equations became the solution of initial–boundary–value problems. A general approach for solving such problems was finally announced in 1997 [1] and was developed further in the works of almost 100 researchers. It is remarkable that these results have motivated the discovery of a new transform method for solving boundary–value problems for linear evolution PDEs with x–derivatives of arbitrary order, as well as for linear elliptic PDEs in two dimensions [2]. This has led to the emergence of a new method in mathematical physics, which is usually referred to as the "Fokas method" or "the unified transform". Several hundred papers have been written using this method, some of which can be found in [http://www.unifiedmethod.azurewebsites.net]. The Fokas method has had a significant impact, from the analysis of boundary value problems for integrable nonlinear PDEs [3] and the introduction of a new method for studying the well posedness of arbitrary nonlinear evolution PDEs [4], to a novel formulation of the classical problem of water waves [5]. This method, which is based on the synthesis, as opposed to the separation of variables [6], unifies and extends several classical branches of mathematics, form the usual transforms to the formulation of Ehrenpreis type integral representations. It is important to note that the solution of any inhomogeneous boundary value problem solved by the usual transforms,suffers from lack of uniform convergence at the boundaries. This serious disadvantage, which renders these representations unsuitable for numerical computations, has not been emphasised in the literature. In contrast, the unified transform yields representations which are uniformly convergent. Thus, it gives new formulae even for such basic problems as the heat equation on the half line, and on a finite interval (see below). Furthermore, it yields effective analytical formulae for several problems for which there do not exist usual transforms [6]. Also, it has given rise to new numerical techniques: for evolution PDEs see, for example, the book "The computation of spectral representations for evolution PDE" by S. Vetra–Carvalh [7], where it is noted that "for linear evolutionary PDEs the numerical implementation of the Fokas method is faster and more accurate than a pseudospectral method"; for elliptic PDEs see, for example, [8]; for other applications see, for example [9].

A pedagogical introduction of the Fokas method is presented in [10]. In an accompanying editorial, the editor of SIAM Review wrote: "Similar to the Fosbury Flop the method of Fokas approached familiar problems from a new direction, providing students and instructors with new insights into linear PDEs".

2 Separation of Variables and Transform Pairs

Until the development of the Fokas method, the most important method for the analytic solution of linear PDEs was the method of *separation of variables*, and the use of an "appropriate" *transform pair*. Consider for example the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}. \tag{2.1}$$

Seeking a separable solution in the form,

$$u(x,t) = X(x)T(t), \tag{2.2}$$

we find

$$XT' = X''T,$$

where prime denotes differentiation.

Thus,

$$\frac{X''}{X} = \frac{T'}{T}. \tag{2.3}$$

Since the LHS of the above equation is a function of x, whereas the RHS is a function of t, it follows that each of the above ratios is a constant, which for convenience we write as $-\lambda^2$, λ arbitrary complex number. Thus, (2.3) yields the two ODE's

$$X''(x) + \lambda^2 X(x) = 0, \tag{2.4}$$

and

$$T'(t) + \lambda^2 T(t) = 0. \tag{2.5}$$

Clearly the representation (2.2) is very limited, however, the intuitive idea is that if we can solve the ODE's (2.4), (2.5), and if we can "sum up" appropriate solutions over λ, then perhaps we can obtain the general solution of the heat equation.

For example, the exponentials $e^{i\lambda x}$ and $e^{-\lambda^2 t}$ are particular solutions of equations (2.4) and (2.5) respectively. Hence, equation (2.2) implies that a particular solution of the heat equation is given by

$$U(\lambda)e^{i\lambda x - \lambda^2 t},$$

where λ is an arbitrary complex constant, and $U(\lambda)$ is an arbitrary function. Clearly, the following expression is also a solution of the heat equation:

$$u(x,t) = \int U(\lambda)e^{i\lambda x - \lambda^2 t} d\lambda, \tag{2.6}$$

where we assume that the above integral makes sense.

It turns out that their exists a general, deep result in analysis known as the Ehrenpreis Principle, which when applied to the particular case of the heat equation, shows that for a well posed problem formulated in a bounded, smooth, convex, domain, the solution can always be written in the form (2.6). However, this result does not provide a systematic way for choosing the above contour, as well as for determining the function $U(\lambda)$.

For the initial value problem of the heat equation, using the Fourier transform pair, it is straighforward to obtain both the relevant contour and the function $U(\lambda)$:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{u}_0(\lambda)d\lambda, \quad -\infty < x < \infty, \quad t > 0, \tag{2.7}$$

where $\hat{u}_o(\lambda)$ is the Fourier transform of $u_0(x)$,

$$\hat{u}_0(\lambda) = \int_{-\infty}^{\infty} e^{-i\lambda x}u_0(x)dx, \quad -\infty < \lambda < \infty.$$

Suppose that $u(x,t)$ satisfies the heat equation (2.1) on the half line,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty, \quad t > 0, \tag{2.8}$$

together with the following initial and boundary conditions:

$$u(x,0) = u_0(x), \quad 0 < x < \infty; \quad u(0,t) = g_0(t), \quad t > 0. \tag{2.9}$$

Usually, this problem is solved via the sine transform pair:

$$\hat{f}_s(\lambda) = \int_0^\infty \sin(\lambda x) f(x) dx, \quad 0 < \lambda < \infty, \tag{2.10}$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) \hat{f}_s(\lambda) d\lambda, \quad 0 < x < \infty. \tag{2.11}$$

By employing the above pair it is straightforward to show that

$$u(x,t) = \frac{2}{\pi} \int_0^\infty \sin(\lambda x) e^{-\lambda^2 t} \left[\int_0^\infty \sin(\lambda \xi) u_0(\xi) d\xi + \lambda \int_0^t e^{\lambda^2 \tau} g_0(\tau) d\tau \right] d\lambda,$$
$$0 < x < \infty, \quad t > 0. \tag{2.12}$$

We first note that this represented is not of the Ehrenpreis form. Second, it is not straightforward to verify that $u(0,t) = g_0(t)$: if we attempt to verify this condition by letting $x = 0$ in the RHS of (2.12) we fail since $\sin 0 = 0$. This implies that we *cannot* exchange the integral with the limit $x \to 0$. In other words, the representation (2.12) is *not* uniformly convergent at $x = 0$ unless $g_0(t) = 0$. This lack of uniform convergence makes the representation (2.12) unsuitable for the *numerical* evaluation of the solution.

It should be emphasised that the above pathology, namely the lack of uniform convergence at the boundary, is a characteristic of *any* solution obtained via the usual transform methods. Indeed, these transforms are defined by considering the *homogeneous* version of the given inhomogeneous problem (see the discussion below). Thus, they construct solutions that are uniformly convergent *only* for homogeneous data.

In addition to the above major disadvantage of the usual transform methods, we also note that in this particular case we were able to "guess" the correct transform. The good news is that there *does* exist a systematic, albeit complicated, way for deriving the appropriate transform pair for a given IBVP. For example, for the case of equations (2.8) and (2.9) one first computes the associated Green's function, namely one solves the following ODE:

$$\frac{\partial^2}{\partial x^2}G(x,\xi,\lambda)+\lambda^2 G(x,\xi,\lambda)=\delta(x-\xi),\quad 0<x<\infty,\quad 0<\xi<\infty,$$

$$G(0,\xi,\lambda)=0,\qquad \lim_{x\to\infty}G(x,\xi,\lambda)=0.$$

Then, one computes the integral of G around an appropriate contour in the complex λ-plane, and this yields the sine transform pair.

The bad news is that for many important IBVPs there *does not* exist an x-transform. For example, there *does not* exist an x-transform for the so-called Stokes equation on the half-line, namely for the equation

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}=0,\quad 0<x<\infty,\quad t>0. \tag{2.13}$$

Indeed, this equation, supplemented with the initial and boundary conditions (2.9), defines an x-spectral problem which is *non-self adjoint*, for which there does *not* exist an appropriate transform.

It should be noted that an evolution PDE in one space variable can also be analysed via a transform in t, which turns out to be the Laplace transform.

Denoting by $\hat{u}_L(s,x)$ the Laplace transform of $u(t,x)$ we find

$$\frac{\partial \hat{u}_L(s,x)}{\partial x}+\frac{\partial^3\hat{u}_L(s,x)}{\partial x^3}+s\hat{u}_L(s,x)=u_0(x),\quad 0<x<\infty. \tag{2.14}$$

One must now solve this third order ODE supplemented with the boundary condition

$$\hat{u}_L(s,0)=\int_0^\infty e^{-st}g_0(t)dt. \tag{2.15}$$

The starting point for solving equation (2.14) is to seek an exponential solution of the homogeneous version of (2.14). Letting $u_L(s,x)=\exp(\Omega(s)x)$ we find that $\Omega(s)$ satisfies the *cubic* equation

$$\Omega(s)^3+\Omega(s)+s=0.$$

In summary, the sine transform pair, in contrast to the Fourier transform pair, has a very limited applicability. Furthermore, the representation (2.12) obtained via this transform is *not* uniformly convergent at $x=0$, and is *not* of the form (2.6). The Laplace transform could in principle be applied to

PDEs involve higher derivatives, but, it has the disadvantage that involves $0 < t < \infty$, and also it requires the analysis of high order nonlinear algebraic equations.

It turns out that there *does* exist the proper analogue of the Fourier transform pair for solving evolution PDEs: in the next section we will define a representation which is both uniformly convergent at $x = 0$ and it is of the form (2.6). Furthermore, analogous representations exist for the solution of a general evolution PDE.

3 The Heat Equation on the Half-Line via the Fokas Method

The new method involves three steps. The first step is identical with the procedure used for the implementation of the usual transforms, whereas the third step involves only algebraic manipulations; the second step requires the use of Cauchy's theorem.

1a. *Given a domain, derive the Global Relation (GR), which is an equation coupling the function and its derivatives on the boundary of the domain.*

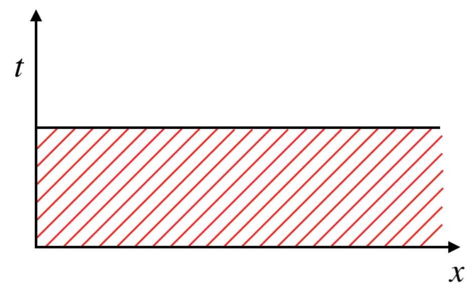

Figure 3.1

For the domain

$$\Omega = \{0 < x < \infty,\ t > 0\}, \tag{3.1}$$

the GR is

$$e^{\lambda^2 t}\hat{u}(-i\lambda, t) = \hat{u_0}(-i\lambda) - \tilde{g}_1(\lambda^2, t) - i\lambda\tilde{g}_0(\lambda^2, t), \quad \Im\lambda \leq 0, \tag{3.2}$$

where

$$\hat{u}(-i\lambda, t) = \int_0^\infty e^{-i\lambda x}u(x,t)dx, \quad t > 0, \quad \Im\lambda \leq 0, \tag{3.3}$$

$$\hat{u_0}(-i\lambda) = \int_0^\infty e^{-i\lambda x}u_0(x)dx, \quad t > 0, \quad \Im\lambda \leq 0, \tag{3.4}$$

$$\tilde{g}_0(\lambda, t) = \int_0^t e^{\lambda\tau}g_0(\tau)d\tau, \quad \tilde{g}_1(\lambda, t) = \int_0^t e^{\lambda\tau}g_1(\tau)d\tau, \quad t > 0, \quad \lambda \in \mathbb{C}, \tag{3.5}$$

with

$$g_1(t) = u_x(0,t), \quad g_0(t) = u(0,t), \quad t > 0. \tag{3.6}$$

Regarding equations (3.3) and (3.4) we note that

$$|e^{-i\lambda x}| = |e^{-i\lambda_R x + \lambda_I x}| = e^{\lambda_I x},$$

thus, this term is bounded as $x \to \infty$, for $\lambda_I < 0$.

The functions $\tilde{g}_0$ and $\tilde{g}_1$ are defined for all complex values of λ, whereas $\hat{u}$ and $\hat{u}_0$ are defined for $\Im\lambda \leq 0$, thus the global relation (3.2) is valid for $\Im\lambda \leq 0$.

Conceptually, the simplest way to derive the global relation is to use the half–Fourier transform, and to follow the same procedure used with the sine transform. Indeed, let the half–Fourier transform of $u(x,t)$ be defined by (3.3).

Then,

$$\begin{aligned}\hat{u}_t &= \int_0^\infty e^{-i\lambda x} u_t dx = \int_0^\infty e^{-i\lambda x} u_{xx} dx \\ &= u_x e^{-i\lambda x}|_0^\infty + i\lambda u e^{-i\lambda x}|_0^\infty - \lambda^2 \hat{u}.\end{aligned}$$

Thus,

$$\hat{u}_t + \lambda^2 \hat{u} = -g_1(t) - i\lambda g_0(t).$$

Hence,

$$(\hat{u} e^{\lambda^2 t})_t = -e^{\lambda^2 t}(g_1(t) + i\lambda g_0(t)),$$

or

$$\hat{u} e^{\lambda^2 t} = \hat{u}_0 - \int_0^t e^{\lambda^2 \tau}[g_1(\tau) + i\lambda g_0(\tau)] d\tau,$$

which is the GR.

2. *Express the solution as an integral in the complex λ-plane involving $\hat{u}_0(-i\lambda)$, as well as the t-transforms of all the relevant boundary values.*

For the heat equation formulated on the half-line, we find

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{u}_0(-i\lambda) d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left[\tilde{g}_1(\lambda^2, t) + i\lambda \tilde{g}_0(\lambda^2, t)\right] d\lambda, \tag{3.7}$$

where the contour ∂D^+ is the boundary of the domain D^+ defined by

$$D^+ = \left\{\Im\lambda \geq 0, \quad \Re\lambda^2 < 0\right\}, \tag{3.8}$$

see figure 3.2.

Indeed, solving the global relation (3.2) for $\hat{u}(-i\lambda, t)$ and then using the inverse Fourier transform formula, we find an expression similar to (3.7) but with the contour of integration along the real line instead of ∂D^+. In order to deform from the real line to ∂D^+ we use Cauchy's theorem and Jordan's Lemma. We first consider the function

$$e^{i\lambda x - \lambda^2 t}\tilde{g}_1(\lambda^2, t) = e^{i\lambda x}\int_0^t e^{-\lambda^2(t-\tau)} g_1(\tau) d\tau,$$

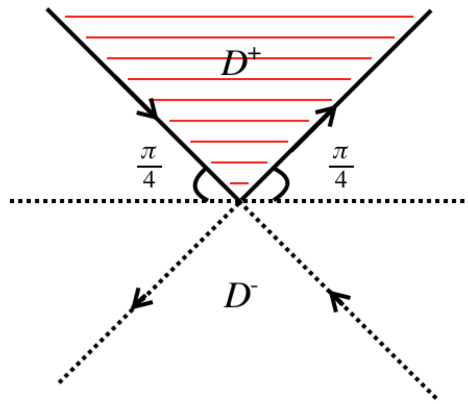

Figure 3.2

which is an analytic function of λ. This function involves the two exponentials

$$e^{i\lambda x} = e^{i\lambda_R x - \lambda_I x}, \quad e^{-\lambda^2(t-\tau)} = e^{-\Re(\lambda^2)(t-\tau) - i\Im(\lambda^2)(t-\tau)},$$

thus since $x \geq 0$ and $t - \tau \geq 0$, the above exponentials are bounded as $\lambda \to \infty$ if λ satisfies $\Im\lambda \geq 0$ and $\Re\lambda^2 \geq 0$. Furthermore, integration by parts implies that the above function is of $O(1/\lambda^2)$ as $\lambda \to \infty$:

$$e^{-\lambda^2 t}\int_0^t e^{\lambda^2\tau} g_1(\tau)d\tau \sim \frac{g_1(t)}{\lambda^2}, \quad \lambda \to \infty.$$

Thus, Cauchy's theorem in the domain bounded by the real line and ∂D^+ implies that the integral of the above function can be deformed from $\mathbb{R}$ to ∂D^+.
The situation is similar with the term $i\lambda \exp[i\lambda x - \lambda^2 t]\tilde{g}_0(\lambda^2, t)$, but now because of the λ factor this function is of $O(1/\lambda)$ as $\lambda \to \infty$, thus we need to supplement Cauchy's theorem with Jordan's lemma.

3. *For given boundary conditions, by employing the global relation as well as certain invariant transformations, eliminate from the integral representation obtained in step 2 the transforms of the unknown boundary values.*

Consider for example the Dirichlet problem of the heat equation formulated on the half line, i.e., equation (2.8) supplemented with the initial and boundary conditions (2.9). In this case, the functions $\hat{u}_0$ and $\tilde{g}_0$ appearing in the global relation (3.2) are known but the functions $\hat{u}$ and $\tilde{g}_1$ are unknown. The global relation is valid for $\Im\lambda \leq 0$, whereas we need $\tilde{g}_1$ for $\lambda \in \partial D^+$, thus we need to compute $\tilde{g}_1$ for $\Im\lambda \geq 0$. We note that the transformation

$\lambda \to -\lambda$ has two crucial properties: first, it maps the domain $\Im\lambda \le 0$ to the domain $\Im\lambda \ge 0$, and also leaves $\tilde{g}_0(\lambda^2, t)$ and $\tilde{g}_1(\lambda^2, t)$ invariant. Using this transformation, the GR yields

$$e^{\lambda^2 t}\hat{u}(i\lambda, t) = \hat{u}_0(i\lambda) - \tilde{g}_1(\lambda^2, t) + i\lambda\tilde{g}_0(\lambda^2, t), \quad \Im\lambda \ge 0. \tag{3.9}$$

Our strategy will be to use equation (3.9) to eliminate $\tilde{g}_1$; in this procedure we ignore the fact that $\hat{u}(i\lambda, t)$ is unknown since it will turn out that its contribution to $u(x, t)$ vanishes. Solving (3.9) for $\tilde{g}_1(\lambda^2, t)$ we find

$$\tilde{g}_1 = i\lambda\tilde{g}_0 + \hat{u}_0(i\lambda) + e^{\lambda^2 t}\hat{u}(i\lambda, t), \quad \Im\lambda \ge 0. \tag{3.10}$$

Replacing in equation (3.7) $\tilde{g}_1$ with the RHS of (3.10) we find

$$u(x,t) =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{u}_0(-i\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left[2i\lambda\tilde{g}_0(\lambda^2, t) + \hat{u}_0(i\lambda)\right]d\lambda. \tag{3.11}$$

The term $\exp(\lambda^2 t)\hat{u}(i\lambda, t)$ gives rise to the term

$$-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x}\hat{u}(i\lambda, t)d\lambda, \quad 0 < x < \infty, \quad t > 0,$$

which vanishes, since both $\exp(i\lambda x)$ and $\hat{u}(i\lambda, t)$ are bounded and analytic in the upper half of the complex λ plane, and furthermore $\hat{u}(i\lambda, t)$ is of $O(1/\lambda)$ as $\lambda \to \infty$:

$$\hat{u}(i\lambda, t) = \int_0^{\infty} e^{i\lambda x}u(x, t)dx \sim -\frac{u(0, t)}{i\lambda}, \quad \lambda \to \infty.$$

Thus, Cauchy's theorem supplemented with Jordan Lemma in the domain D^+ imply the desired result.

Remarks

(a) Suppose that the heat equation is valid for $0 < t < T$. Let

$$\tilde{g}_0(\lambda) = \tilde{g}_0(\lambda, T), \quad \tilde{g}_1(\lambda) = \tilde{g}_1(\lambda, T). \tag{3.12}$$

Then, equation (3.7) is equivalent to the equation

$$u(x,t) =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{u}_0(-i\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left[\tilde{g}_1(\lambda^2)+i\lambda\tilde{g}_0(\lambda^2)\right]d\lambda. \tag{3.13}$$

Indeed, the RHS of equation (3.7) and the RHS of equation (3.13) differ by the term

$$\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x}\left[\int_t^T e^{\lambda^2(\tau-t)}g_1(\tau)d\tau + i\lambda\int_t^T e^{\lambda^2(\tau-t)}g_0(\tau)d\tau\right]d\lambda,$$

and Cauchy's theorem supplemented with Jordan's lemma imply that the above term vanishes.

Similarly, equation (3.11) is equivalent to the equation

$$u(x,t) =$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{u}_0(-i\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left[2i\lambda\tilde{g}_0(\lambda^2)+\hat{u}_0(i\lambda)\right]d\lambda. \tag{3.14}$$

This equation is of the Ehrenpreis form (2.6). Actually, the Fokas method always yields representations of this form. The advantage of (3.14) is that the only (x,t) dependence of the RHS of this equation is of the form $e^{i\lambda x-\lambda^2 t}$, thus it immediately follows that the function u defined in (3.14) satisfies the heat equation. On the other hand, (3.7) is consistent with causality, since the function $u(x,t)$ cannot depend on the values of $g_0(\tau)$ for $\tau > t$.

(b) In deriving (3.7), the real line was deformed to ∂D^+. This deformation is always possible *before* using the global relation. However, *after* using the global relation we introduce $\hat{u}_0$ and then it is *not* always possible to return to the real axis. Actually, the cases where there *do* exist usual transforms, are precisely the cases where this "return" is possible.

In the particular case of (3.7), we note that $\hat{u}_0(i\lambda)$ is bounded and analytic in the upper half of the complex λ plane, thus it *is* possible to return to the real axis:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\left[\hat{u}_0(-i\lambda)-\hat{u}_0(i\lambda)\right]d\lambda - \frac{i}{\pi}\int_{-\infty}^{\infty}\lambda e^{i\lambda x-\lambda^2 t}\tilde{g}_0(\lambda^2,t)d\lambda.$$

Splitting the integral along $\mathbb{R}$ to an integral from $-\infty$ to 0 plus an integral from 0 to ∞, and letting $\lambda \to -\lambda$ in the former integral we obtain the representation obtaned in section 2 via the sine transform. An easier way to obtain this representation is to recall that the global relation together with the equation obtained from the global relation after replacing λ with $-\lambda$ are the following equations:

$$e^{\lambda^2 t}\hat{u}(-i\lambda,t)=\hat{u}_0(-i\lambda)-\tilde{g}_1-i\lambda\tilde{g}_0,\quad \Im\lambda\leq 0,$$
$$e^{\lambda^2 t}\hat{u}(i\lambda,t)=\hat{u}_0(i\lambda)-\tilde{g}_1+i\lambda\tilde{g}_0,\quad \Im\lambda\geq 0. \tag{3.15}$$

If λ is real, then both these equations are valid. Hence if g_0 is given, we subtract equations (3.15) and we obtain the equation for the sine transform of $u(x,t)$. Similarly, if $u_x(0,t)$ is given, we add equations (3.15) and we obtain

$$e^{\lambda^2 t}\hat{u}_c(\lambda,t)=\hat{u}_{0_c}(-i\lambda)-\tilde{g}_1(\lambda^2,t),\quad \lambda\in\mathbb{R},$$

where $\hat{u}_c$ and $\hat{u}_{0_c}$ denote the cosine transform of $u(x,t)$ and $u_0(x)$ respectively, namely:

$$\hat{u}_c(\lambda,t)=\int_0^\infty \cos(\lambda x)u(x,t)dx,\quad \hat{u}_{0c}(\lambda)=\int_0^\infty \cos(\lambda x)u_0(x)dx.$$

(c) Equation (3.14) immediately implies that $u(x,t)$ satisfies the heat equation. Furthermore, evaluating (3.14) at $t=0$ we find

$$u(x,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\lambda x}\hat{u}_0(-i\lambda)d\lambda-\frac{1}{2\pi}\int_{\partial D^+}e^{i\lambda x}\hat{u}_0(i\lambda)d\lambda,\quad x>0.$$

Jordan's lemma implies that the second integral in the above expression vanishes and hence by recalling the definition of $\hat{u}_0(-i\lambda)$ and employing the inverse Fourier transform formula we find $u(x,0)=u_0(x)$.

Evaluating (3.14) at $x=0$ we find

$$u(0,t)=$$
$$\frac{1}{2\pi}\left[\int_{-\infty}^{\infty}e^{-\lambda^2 t}\hat{u}_0(-i\lambda)d\lambda-\int_{\partial D^+}e^{-\lambda^2 t}\hat{u}_0(i\lambda)d\lambda\right]-\frac{1}{2\pi}\int_{\partial D^+}2i\lambda e^{-\lambda^2 t}\tilde{g}_0(\lambda^2)d\lambda. \tag{3.16}$$

By deforming the second integral to the real axis and then replacing λ with $-\lambda$ we find that the first two terms in the RHS of (3.16) cancel. Furthermore, letting $i\lambda^2=l$ in the last integral in the RHS of (3.16) we find

$$u(0,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ilt}\Big(\int_0^T e^{-il\tau}g_0(\tau)d\tau\Big)dl=g_0(t).$$

Numerical Evaluations[11]

For the simple cases when the transforms of the given data can be computed explicitly, the numerical evaluation of the solution obtained by the Fokas method reduces to the computation of a *single* integral in the complex λ-plane. Using simple contour deformations, it is possible to obtain an integrand which decays exponentially as $\lambda \to \infty$.

Example Consider the heat equation on the half line with

$$u(x,0) = e^{-a^2 x}, \quad u(0,t) = \cos(bt), \quad a, b \text{ real constants.}$$

Then,

$$\hat{u}_0(-i\lambda) = \int_0^\infty e^{-i\lambda x - a^2 x} d\lambda = \frac{1}{i\lambda + a^2},$$

$$\tilde{g}_0(\lambda, t) = \int_0^t e^{\lambda\tau} \cos(b\tau) d\tau = \frac{1}{2}\left[\frac{e^{(\lambda+ib)t} - 1}{\lambda + ib} + \frac{e^{(\lambda-ib)}t - 1}{\lambda - ib}\right].$$

Hence (3.7) becomes

$$2\pi u(x,t) = \int_{-\infty}^{\infty} \frac{e^{i\lambda x - \lambda^2 t}}{i\lambda + a^2} d\lambda$$

$$- \int_{\partial D^+} e^{i\lambda x - \lambda^2 t}\left[\frac{1}{-i\lambda + a^2} + \frac{i\lambda}{\lambda + ib}\left(e^{(\lambda^2+ib)t} - 1\right) + \frac{i\lambda}{\lambda - ib}\left(e^{(\lambda^2-ib)t} - 1\right)\right] d\lambda.$$

The term $\exp(i\lambda x)$ in the integrand of the second integral decays as $\lambda \to \infty$, but the term $\exp(-\lambda^2 t)$ oscillates. However, if we deform ∂D^+ to a contour L between the real line and ∂D^+, then we achieve exponential decay in both $\exp(i\lambda x)$ and $\exp(-\lambda^2 t)$:

$$2\pi u(x,t) = \int_L \left\{e^{i\lambda x - \lambda^2 t}\left[\frac{1}{i\lambda + a^2} + \frac{1}{i\lambda - a^2}\right] + i\lambda e^{i\lambda x}\left[\frac{e^{ibt} - e^{-\lambda^2 t}}{\lambda + ib} + \frac{e^{-ibt} - e^{-\lambda^2 t}}{\lambda - ib}\right]\right\} d\lambda,$$

where L depicted in Figure 3.3.

The above integral can be computed using the demand of MATLAB. With the command we have stated in the homework

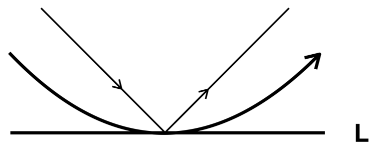

Figure 3.3

4 The Heat Equation on the Finite-Interval via the Fokas Method

We now solve the heat equation formulated on the finite interval

$$0 < x < L. \tag{4.1}$$

In what follows, we implement steps 1, 2, and 3, of section 3.

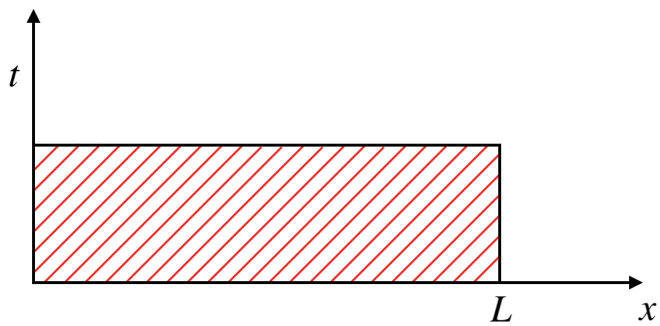

Figure 4.1

Step 1. In analogy with equation (3.2) we now have

$$e^{\lambda^2 t}\hat{u}(-i\lambda,t) = \hat{u}_0(-i\lambda) - \tilde{g}_1(\lambda^2,t) - i\lambda\tilde{g}_0(\lambda^2,t) + e^{-i\lambda L}\left[\tilde{h}_1(\lambda^2,t) + i\lambda\tilde{h}_0(\lambda^2,t)\right],$$

$$\lambda \in \mathbb{C}, \tag{4.2}$$

where $\hat{u}$ and $\hat{u}_0$ are the finite Fourier transforms of $u(x,t)$ and $u_0(x)$, defined by

$$\hat{u}(-i\lambda,t) = \int_0^L e^{-i\lambda x}u(x,t)dx, \quad \hat{u}_0(-i\lambda) = \int_0^L e^{-i\lambda x}u_0(x)dx, \quad \lambda \in \mathbb{C}, \tag{4.3}$$

$\tilde{g}_1$, $\tilde{g}_0$ are defined in (3.5) and $\tilde{h}_1$, $\tilde{h}_0$ are defined by

$$\tilde{h}_0(\lambda, t) = \int_0^t e^{\lambda\tau} h_0(\tau) d\tau, \quad \tilde{h}_1(\lambda, t) = \int_0^t e^{\lambda\tau} h_1(\tau) d\tau, \quad t > 0, \ \lambda \in \mathbb{C}, \tag{4.4}$$

with $h_0(t) = u(L, t)$, $h_1(t) = u_x(L, t)$, $t > 0$.

In order to derive (4.2) we consider the finite Fourier Transform of $u(x, t)$ defind in (4.3). Then,

$$\hat{u}_t = \int_0^L e^{-i\lambda x} u_{xx} dx = u_x e^{-i\lambda x}\Big|_0^L + i\lambda u e^{-i\lambda x}\Big|_0^L - \lambda^2 \hat{u}.$$

Thus,

$$\hat{u}_t + \lambda^2 \hat{u} = -g_1(t) - i\lambda g_0(t) + e^{-i\lambda L}(h_1(t) + \lambda h_0(t)).$$

Hence

$$(\hat{u} e^{\lambda^2 t})_t = -e^{\lambda^2 t}(g_1(t) + i\lambda g_0(t)) + e^{\lambda^2 t - i\lambda L}(h_1(t) + i\lambda h_0(t)),$$

which upon integration implies (4.2)

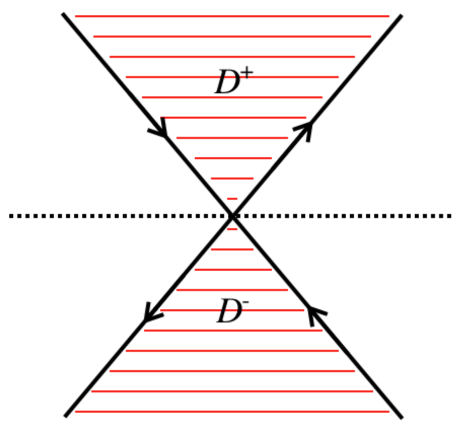

Figure 4.2

Step 2. Solving (4.2) for $\hat{u}(-i\lambda, t)$, using the inverse Fourier transform formula, well as deforming from R to ∂D^+ in the integral involving $\tilde{g}_1, \tilde{g}_0$, and from R to ∂D^- in the integral involving $\tilde{h}_1, \tilde{h}_0$, we find

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t} \hat{u}_0(-i\lambda) d\lambda - \frac{1}{2\pi} \int_{\partial D^+} e^{i\lambda x - \lambda^2 t} \left[\tilde{g}_1 + i\lambda \tilde{g}_0\right] d\lambda$$

$$-\frac{1}{2\pi}\int_{\partial D^-} e^{-i\lambda(L-x)-\lambda^2 t}\left[\tilde{h}_1 + i\lambda\tilde{h}_0\right] d\lambda, \tag{4.5}$$

where D^- is the reflection of D^+ with respect to the real axis and D^- is to the left of the increasing direction of ∂D^-, see figure 4.2.

Step 3. The transformation $\lambda \to -\lambda$ together with the global relation (4.2) yields two equations. Since there exist four unknown boundary values (two at each end of the domain), we require two boundary conditions. However, we cannot assign these conditions in an arbitrary manner. It can be shown that the terms arising from $\hat{u}(\pm i\lambda, t)$ are bounded as $\lambda \to \infty$ in the relevant domains D^+ and D^-, if and only if one boundary condition is prescribed at each end of the domain.

Example

$$u(0,t) = g_0(t), \quad u(L,t) = h_0(t).$$

The global relation (4.2) can be written in the form

$$e^{\lambda^2 t}\hat{u}(-i\lambda, t) = G(\lambda, t) - \tilde{g}_1 + e^{-i\lambda L}\tilde{h}_1, \tag{4.6}$$

where the known function G is defined by

$$G(\lambda, t) = \hat{u}_0(-i\lambda) - i\lambda\tilde{g}_0(\lambda^2, t) + i\lambda e^{-i\lambda L}\tilde{h}_0(\lambda^2, t). \tag{4.7}$$

Letting $\lambda \mapsto -\lambda$ in (4.6), and recalling that $\tilde{g}_1$ and $\tilde{h}_1$ are invariant with respect to $\lambda \mapsto -\lambda$, we obtain

$$e^{\lambda^2 t}\hat{u}(i\lambda, t) = G(-\lambda, t) - \tilde{g}_1 + e^{i\lambda L}\tilde{h}_1. \tag{4.8}$$

Solving equations (4.6) and (4.8) for $\tilde{g}_1$ and $\tilde{h}_1$, we find

$$\tilde{g}_1 =$$

$$\frac{1}{e^{i\lambda L} - e^{-i\lambda L}}\left\{e^{i\lambda L}G(\lambda, t) - e^{-i\lambda L}G(-\lambda, t) + e^{\lambda^2 t}\left[e^{-i\lambda L}\hat{u}(i\lambda, t) - e^{i\lambda L}\hat{u}(-i\lambda, t)\right]\right\}, \tag{4.9}$$

$$\tilde{h}_1 = \frac{1}{e^{i\lambda L} - e^{-i\lambda L}}\left\{G(\lambda, t) - G(-\lambda, t) + e^{\lambda^2 t}\left[\hat{u}(i\lambda, t) - \hat{u}(-i\lambda, t)\right]\right\}. \tag{4.10}$$

We next substitute $\tilde{g}_1$ and $\tilde{h}_1$ in (4.5). We claim that the terms involving $\hat{u}(\pm i\lambda, t)$ yield a zero contribution. Indeed, since this is a well-posed BVP, the relevalt terms are bounded as $\lambda \to \infty$. Let us verify this explicitly; the term in $\tilde{g}_1$ involves the following contribution from $\hat{u}(\pm i\lambda, t)$:

$$\frac{e^{-i\lambda L}\hat{u}(i\lambda,t) - e^{i\lambda L}\hat{u}(-i\lambda,t)}{e^{i\lambda L} - e^{-i\lambda L}}.$$

Since $\Im\lambda \geq 0$, $e^{-i\lambda L}$ grows, and then the above expression, as $\lambda \to \infty$, becomes

$$-\hat{u}(i\lambda,t) + e^{i\lambda L}\int_0^L e^{i\lambda(L-x)}u(x,t)dx,$$

which is clearly bounded as $\lambda \to \infty$ with $\Im\lambda \geq 0$. Similarly the term in $\tilde{h}_1$ involves the following contribution from $\hat{u}(\pm i\lambda, t)$:

$$\frac{\hat{u}(-i\lambda,t) - \hat{u}(i\lambda,t)}{e^{i\lambda L} - e^{-i\lambda L}},$$

which as $\lambda \to \infty$, $\Im\lambda \leq 0$, simplifies to the expression

$$\int_0^L e^{-i\lambda(L-x)}u(x,t)dx - e^{-i\lambda L}\hat{u}(i\lambda,t),$$

which is clearly bounded as $\lambda \to \infty$, $\Im\lambda \leq 0$.

We also note that the zeros of $\exp(i\lambda L) - \exp(-i\lambda L)$ occur on the real axis, and hence are outside D except for $\lambda = 0$ which is a removable singularity, since

$$\left[e^{-i\lambda L}\hat{u}(-i\lambda,t) - e^{i\lambda L}\hat{u}(i\lambda,t)\right]_{\lambda=0} = \left[\hat{u}(-i\lambda,t) - \hat{u}(i\lambda,t)\right]_{\lambda=0} = 0.$$

Thus, (4.5) becomes

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x - \lambda^2 t}\hat{u}_0(\lambda)d\lambda$$

$$-\frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\left[i\lambda\tilde{g}_0(\lambda^2,t) + \frac{e^{i\lambda L}G(\lambda,t) - e^{-i\lambda L}G(-\lambda,t)}{e^{i\lambda L} - e^{-i\lambda L}}\right]d\lambda$$

$$-\frac{1}{2\pi}\int_{\partial D^-} e^{i\lambda x-\lambda^2 t}\left[i\lambda\tilde{h}_0(\lambda^2,t) + \frac{G(\lambda,t) - G(-\lambda,t)}{e^{i\lambda L} - e^{-i\lambda L}}\right]d\lambda, \qquad (4.11)$$

where ∂D^+ and ∂D^- are detected in Figure 4.2.

Remarks 4.1.

1. It is possible to deform ∂D^+ and ∂D^- back to the real axis and then using the residue theorem the usual sine-sine solution can be rederived. A simpler way to obtain the usual solution representation is to subtract (4.6), (4.8):

$$2ie^{\lambda^2 t}\int_0^L \sin(\lambda x)u(x,t)dx = (e^{i\lambda L}-e^{-i\lambda L})\tilde{h}_1(\lambda^2,t)+G(-\lambda,t)-G(\lambda,t). \tag{4.12}$$

The unknown function $\tilde{h}_1$ can be eliminated by evaluating the above equation at those values of λ for which the coefficient of $\tilde{h}_1$ vanishes:

$$e^{i\lambda L}-e^{-i\lambda L}=0, \quad \lambda=\frac{n\pi}{L}, \quad n=0,1,2,\ldots.$$

Hence (4.12) becomes

$$2ie^{\left(\frac{n\pi}{L}\right)^2 t}\int_0^L \sin\left(\frac{n\pi x}{L}\right)u(x,t)dx = G\left(-\frac{n\pi}{L},t\right)-G\left(\frac{n\pi}{L},t\right),$$

and then the usual representation follows using the following transform pair:

$$f_n=\frac{2}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)f(x)dx, \quad n=1,2,\ldots$$
$$f(x)=\sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi x}{L}\right).$$

2. The function $e^{i\lambda L}-e^{-i\lambda L}$, which appears in (4.11), has simple poles at the points $n\pi/L$ which occur on the real axis. Thus, the classical representational is formulated on the "worst" part of the complex plane. Perhaps this is related with the fact that this classical representation is not uniformly convergent at $x=0$ and $x=L$.

3. The numerical implementation of the Fokas method for evolution PDEs on the finite interval is discussed in [12].

   Example

$$u_x(0,t)-\gamma u(0,t)=g_R(t), \quad u_x(L,t)=0, \quad \gamma>0.$$

   The classical representational involves a series area, over $\{\lambda\}_1^{\infty}$, where are the real zeros of the transcendental equation

$$\Delta(\lambda)=(i\lambda-\gamma)e^{-i\lambda L}-(i\lambda+\gamma)e^{i\lambda L}. \tag{4.13}$$

This series is not uniformly convergent at $x = 0$ and $x = L$.

On the other hand, the Fokas method yields a solution which is similar to (4.11):

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\lambda x-\lambda^2 t}\hat{u}_0(-i\lambda)d\lambda - \frac{1}{2\pi}\int_{\partial D^+} e^{i\lambda x-\lambda^2 t}\frac{\tilde{g}(\lambda)}{\Delta(\lambda)}d\lambda$$

$$-\frac{1}{2\pi}\int_{\partial D^-} e^{i\lambda x-\lambda^2 t}\frac{\tilde{h}(\lambda)}{\Delta(\lambda)}d\lambda, \tag{4.14}$$

where $\hat{u}_0(-i\lambda)$ is the Fourier transform of $u_0(x)$, $\Delta(\lambda)$ is defined by (4.13), ∂D^+ and ∂D^- are defined as in (4.5) and the transforms $\tilde{g}$, $\tilde{h}$ are explicitly given in and terms of $\hat{u_0}(\pm i\lambda)$, and or g_R which is the t–transforms of g_R:

$$\tilde{g}(\lambda) = 2i\lambda e^{-i\lambda L}\tilde{g}_R(\lambda^2) - (i\lambda+\gamma)(e^{i\lambda L}\hat{u}_0(-i\lambda) + e^{-i\lambda L}\hat{u}_0(i\lambda),$$

and

$$\tilde{h}(\lambda) = 2i\lambda\tilde{g}_R(\lambda^2) - (i\lambda-\gamma)\hat{u}_0(-i\lambda) - (i\lambda+\gamma)\hat{u}_0(i\lambda).$$

5 Elliptic equation in the Interior of a Convex Polygon

The most important elliptic PDEs are the Laplace, the modified Helmholtz and the Helmholtz equations. The Laplace equation is:

$$u_{xx} + u_{yy} = 0. \tag{5.1}$$

If u satisfies the Laplace equation (5.1), then u is called a *harmonic* function. Traditionally, harmonic functions are associated with the real and imaginary parts of an analytic function. However, there is an alternative direct way to associate harmonic and analytic functions:

the function $u(x, y)$, which may be complex, is harmonic if and only if u_z is analytic.

Indeed, if u_z is analytic then $u_{z\bar{z}} = 0$, i.e., u is harmonic; the inverse is also true.

The Global Relation

Recall that the first step of the Fokas method consists of deriving the *global relation.*

For elliptic PDEs involving second order derivatives, we need *two* global relations. However, if we assume that u is real, then the second global relation can be obtained from the first via complex conjugation.

The simplest way to derive a global relation is to consider the formal adjoint of the Laplace equation, which is itself,

$$v_{xx} + v_{yy} = 0. \tag{5.2}$$

Multiplying equations (5.1) and (5.2) by v and u respectively, and then subtracting the resulting equations we find

$$(vu_x - uv_x)_x + (vu_y - uv_y)_y = 0.$$

Letting $v = \exp(-i\lambda x + \lambda y)$, which is a particular solution of (5.2) for any complex constant λ, we find the family of conservation laws

$$\left[e^{-i\lambda x+\lambda y}(u_x + i\lambda u)\right]_x + \left[e^{-i\lambda x+\lambda y}(u_y - \lambda u)\right]_y = 0, \quad \lambda \in \mathbb{C}. \tag{5.3}$$

The exponential $\exp(i\lambda x+\lambda y)$ provides an other particular solution of (5.2), and this yields

$$\left[e^{i\lambda x+\lambda y}(u_x - i\lambda u)\right]_x + \left[e^{i\lambda x+\lambda y}(u_y - \lambda u)\right]_y = 0, \quad \lambda \in \mathbb{C}. \tag{5.4}$$

We note that if u is real, then equation (5.4) can be obtained from (5.3) by taking the complex conjugate and then replacing in the resulting equation λ by $\bar{\lambda}$. This procedure is called *Schwartz conjugation.*

Suppose that the Laplace equation is valid in the domain Ω. Then, equations (5.3) and (5.4) together with Green's theorem, imply the following global relations:

$$\int_{\partial\Omega} e^{-i\lambda x+\lambda y}\left[(u_x+i\lambda u)dy-(u_y-\lambda u)dx\right]=0, \quad \lambda\in\mathbb{C}, \tag{5.5}$$

and

$$\int_{\partial\Omega} e^{i\lambda x+\lambda y}\left[(u_x-i\lambda u)dy-(u_y-\lambda u)dx\right]=0, \quad \lambda\in\mathbb{C}, \tag{5.6}$$

where $\partial\Omega$ denotes the boundary of Ω.

The most well known boundary value problems for elliptic PDEs are either the *Dirichlet problem* where u is prescribed on the boundary, or the *Neumann problem* where the normal derivative, denoted by u_ω, is prescribed on the boundary.

In order to rewrite the global relations in terms of u and u_ω, we parameterize the boundary $\partial\Omega$ in terms of its arclength which we denote by s. Then, if u_T denotes the derivative of u along the tangent to $\partial\Omega$, and u_ω denotes the derivative of u normal to u_T in the outward direction, then differentiating $u(x(s),y(s))$ we find

$$u_x dx+u_y dy=u_T ds. \tag{5.7}$$

Since the infinitesimal vector $(dy,-dx)$ is normal to the infinitesimal vector (dx,dy), we find

$$u_x dy-u_y dx=u_\omega ds. \tag{5.8}$$

Thus, we can rewrite equations (5.5) and (5.6) in terms of u and u_ω:

$$(u_x+i\lambda u)dy-(u_y-\lambda u)dx=u_\omega ds+\lambda u(dx+idy).$$

Hence, the global relation (5.5) becomes

$$\int_{\partial\Omega} e^{-i\lambda x+\lambda y}\left[u_\omega+\lambda u\Big(\frac{dx}{ds}+i\frac{dy}{ds}\Big)\right]ds=0. \tag{5.9}$$

Similarly

$$\int_{\partial\Omega} e^{i\lambda x+\lambda y}\left[u_\omega+\lambda u\Big(\frac{dx}{ds}-i\frac{dy}{ds}\Big)\right]ds=0. \tag{5.10}$$

Letting

$$z = x + iy, \quad \bar{z} = x - iy, \tag{5.11}$$

equations (5.9) and (5.10) become

$$\int_{\partial\Omega} e^{-i\lambda z}\big(u_\omega + \lambda u \frac{dz}{ds}\big) ds = 0, \tag{5.12}$$

and

$$\int_{\partial\Omega} e^{i\lambda \bar{z}}\big(u_\omega + \lambda u \frac{d\bar{z}}{ds}\big) ds = 0. \tag{5.13}$$

A Polygonal Domain

Let Ω be the interior of the polygonal domain specified by the complex numbers z_1, z_2, $\ldots$,z_n, $z_{n+1} = z_1$, see figure 5.1.

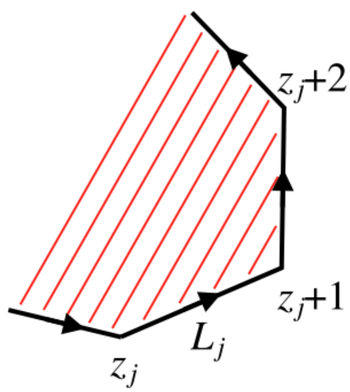

Figure 5.1

Let L_j denote the side (z_j, z_{j+1}).

Then, the global relation (5.12) becomes

$$\sum_{j=1}^{n} \hat{W}_j + \lambda \sum_{j=1}^{n} \hat{D}_j = 0, \quad \lambda \in \mathbb{C}, \tag{5.14}$$

where $\{W_j\}_1^n$ denote the transforms of the Neumann boundary values and $\{D_j\}_1^n$ denote the transforms of the Dirichlet boundary values:

$$\hat{W}_j = \int_{z_j}^{z_{j+1}} e^{-i\lambda z} u_{w_j} ds, \quad j = 1, 2, \ldots, n, \quad \lambda \in \mathbb{C} \tag{5.15}$$

and

$$\hat{D}_j = \int_{z_j}^{z_{j+1}} e^{-i\lambda z} u_j \frac{dz}{ds} ds, \quad j = 1, 2, \ldots, n, \quad \lambda \in \mathbb{C}. \tag{5.16}$$

If u is real, then instead of analysing the global relation (5.13), we can analyse the complex conjugate of equation (5.12). Thus, for real u, equation (5.12) and its complex conjugate provide *two equations* for n *unknown functions*, since for a well posed problem only one boundary condition is given on each side. This situation appears ominus, however in equation (5.12) the complex constant λ is *arbitrary*, thus in this sense equation (5.12) contains "infinitely many" equations. It turns out that this observation provides a most efficient way for the numerical integration of this problem.

Approximate Global Relations

The numerical solution of the global relations for determining the unknown boundary values involves the following two steps [13]-[15]:

1. Expand the function $[u_j]_1^n$ and $\left[\frac{\partial u_j}{\partial w}\right]_1^u$ in terms of N basis functions denoted by $\{S_l(t)\}_0^{N-1}$:

$$u_j(t) \approx \sum_{l=0}^{N-1} a_l^j S_l(t), \quad \frac{\partial u_j(t)}{\partial \omega} \approx \sum_{l=0}^{N-1} b_l^j S_l(t), \quad j = 1, 2, \ldots, n.$$

A convenient such basis is given by the Legendre polynomials of order l, denote by P_l.

Let $\hat{S}_l(\lambda)$ denote the Fourier transform of $S_l(t)$, namely

$$\hat{S}_l(\lambda) = \int_{-1}^{1} e^{-i\lambda t} S_l(t) dt, \quad \lambda \in \mathbb{C}. \tag{5.17}$$

For the Legendre polynomials the relevant Fourier transform can be computed explicitly,

$$\int_{-1}^{1} e^{-i\lambda t} P_l(t) dt = i \sum_{k=0}^{l} \frac{(l+k)!}{(l-k)!k!} \left[\frac{(-1)^{l+k} e^{i\lambda} - e^{-i\lambda}}{(2i\lambda)^{k+1}} \right]. \tag{5.18}$$

Then, the global relation and its complex conjugate yield two equations involving the constants a_l^j and b_l^j. By evaluating these equations at appropriately chosen values of λ called *collocation points*, we can solve for the unknown coefficients.

Example

Consider the Laplace equation in the interior of the square with corners

$$z_1 = -1 + i, \quad z_2 = -1 - i, \quad z_3 = 1 - i, \quad z_4 = 1 + i.$$

Then, the global relation (5.12) involves the following terms:

$$\begin{aligned}
\hat{u}_1(\lambda) &= e^{i\lambda} \int_1^{-1} e^{\lambda y} \left[u_x^{(1)} + i\lambda u^{(1)} \right] dy, \\
\hat{u}_2(\lambda) &= e^{-\lambda} \int_{-1}^{1} e^{-i\lambda x} \left[-u_y^{(2)} + \lambda u^{(2)} \right] dx, \\
\hat{u}_3(\lambda) &= e^{-i\lambda} \int_{-1}^{1} e^{\lambda y} \left[u_x^{(3)} + i\lambda u^{(3)} \right] dy, \\
\hat{u}_4(\lambda) &= e^{\lambda} \int_1^{-1} e^{-i\lambda x} \left[-u_y^{(4)} + \lambda u^{(4)} \right] dx. \qquad (5.19)
\end{aligned}$$

Let

$$\hat{W}(\lambda) = \int_{-1}^{1} e^{\lambda t} W(t) dt, \quad \hat{D}(\lambda) = \int_{-1}^{1} e^{\lambda t} D(t) dt, \quad \lambda \in \mathbb{C}, \qquad (5.20)$$

where $W(t)$ and $D(t)$ denote Neumann and Dirichlet boundary values respectively. Then,

$$\begin{aligned}
\hat{u}_1(\lambda) &= -e^{i\lambda} \left[\hat{W}_1(\lambda) + i\lambda \hat{D}_1(\lambda) \right], \\
\hat{u}_2(\lambda) &= e^{-\lambda} \left[\hat{W}_2(-i\lambda) + \lambda \hat{D}_2(-i\lambda) \right], \\
\hat{u}_3(\lambda) &= e^{-i\lambda} \left[\hat{W}_3(\lambda) + i\lambda \hat{D}_3(\lambda) \right], \\
\hat{u}_4(\lambda) &= e^{\lambda} \left[\hat{W}_4(-i\lambda) - \lambda \hat{D}_4(-i\lambda) \right]. \qquad (5.21)
\end{aligned}$$

The approximate global relation yields

$$\hat{u}_1(\lambda) + \hat{u}_2(\lambda) + \hat{u}_3(\lambda) + \hat{u}_4(\lambda) = 0, \quad \lambda \in \mathbb{C}, \qquad (5.22)$$

where

$$\hat{u}_1(\lambda) \approx -e^{i\lambda} \sum_{l=0}^{N-1} \left[i\lambda a_1^l \hat{P}_l(\lambda) + b_1^l \hat{P}_l(\lambda) \right],$$

$$\hat{u}_2(\lambda) \approx e^{-\lambda} \sum_{l=0}^{N-1} \left[\lambda a_2^l \hat{P}_l(-i\lambda) + b_2^l \hat{P}_l(-i\lambda) \right],$$

$$\hat{u}_3(\lambda) \approx e^{-i\lambda} \sum_{l=0}^{N-1} \left[i\lambda a_3^l \hat{P}_l(\lambda) + b_3^l \hat{P}_l(\lambda) \right],$$

$$\hat{u}_4(\lambda) \approx e^{\lambda} \sum_{l=0}^{N-1} \left[-\lambda a_4^l \hat{P}_l(-i\lambda) + b_4^l \hat{P}_l(-i\lambda) \right]. \tag{5.23}$$

2. For a given side, choose λ in such a way that for the given side we obtain the usual Fourier transform (FT) of the Legendre functions, whereas the contribution from the remaining sides vanishes as $\lambda \to \infty$ It turns out that for a *convex* polygon such a choice is always possible).

- side 1. Multiply (5.22) by $e^{-i\lambda}$ and then let $\lambda = -i\rho$, $\rho > 0$.

We find the following forms for $\hat{W}_j$ (and similarly for $\left[\hat{D}_j\right]_1^4$:

$$\hat{W}_1(-i\rho), \quad e^{i\rho} e^{-\rho} \hat{W}_2(-\rho), \quad e^{-2\rho} \hat{W}_3(-i\rho), \quad e^{-i\rho} e^{-\rho} \hat{W}_4(-\rho).$$

The first terms involve the FT, whereas the remaining terms vanish as $\rho \to \infty$. This is obvious for the third term, whereas the second and the fourth terms involve the integral

$$\int_{-1}^{1} e^{-\rho(1+t)} w(t) dt;$$

since $-1 < t < 1$, it follows that $1 + t > 0$, thus $\exp[-\rho(1 + t)]$ vanishes as $\rho \to \infty$.

- side 2. Multiply by e^{λ} and then let $\lambda = -\rho$, $\rho > 0$.

- side 3. Multiply by $e^{i\lambda}$ and then let $\lambda = i\rho$, $\rho > 0$.

- side 4. Multiply by $e^{-\lambda}$ and then let $\lambda = \rho$, $\rho > 0$.

For ρ we can use the discrete values $\rho = \frac{R}{M} m$, $m = 1, 2, \ldots, M$, $R > 0$, where R/M determines how close are the collocation points.

It is found numerically [15] that the following rules for low condition number:

$$\frac{R}{m} \geq 2, \quad M \geq Nn.$$

The above numerical technique can be viewed as the counterpart in the complex Fourier plane of the boundary integral method (which is formulated in the physical plane).

6 Modified Helmholtz and Helmholtz Equations

For the other two basic elliptic equations the situational is similar. In particular, for the modified Helmholtz equation,

$$u_{xx} + u_{yy} - k^2 u = 0, \quad (x,y) \in \Omega; \quad k > 0, \tag{6.1}$$

the global relation is given by

$$\int_{\partial\Omega} e^{\frac{ik}{2}\left[\frac{\bar{z}(t)}{\lambda} - \lambda z(t)\right]} \left[u_\omega + \frac{ku}{2}\left(\lambda \frac{dz(t)}{dt} + \frac{1}{\lambda}\frac{d\bar{z}(t)}{dt}\right)\right] dt = 0, \quad k > 0, \quad \lambda \in \mathbb{C}\backslash\{0\}. \tag{6.2}$$

For the case that Ω is the interior of the polygon with corners at $z_1, z_2, \ldots, z_n$, equation (6.2) becomes

$$\sum_{j=1}^{n} \hat{u}_j(\lambda) = 0, \quad \lambda \in \mathbb{C}\backslash\{0\}, \tag{6.3}$$

where $\hat{u}_j(\lambda)$ is defined by

$$\hat{u}_j(\lambda) = \int_{z_j}^{z_{j+1}} e^{-i\frac{k}{2}(\lambda z - \frac{\bar{z}}{\lambda})} \left[\left(u_z + i\frac{k}{2}\lambda u \right) dz - (u_{\bar{z}} + \frac{k}{2i\lambda} u) d\bar{z} \right], \quad j = 1, \cdots, n. \tag{6.4}$$

A second global relation is obtained from equation (6.2) by replacing λ with $1/\lambda$. If u is real, we can obtain the second global relation by taking the Schwarz conjugate of equation (6.2).

Remark Recall that for evolution PDE's the second step of the Fokas method involves the derivation of an integral representation, defined in the complex plane which depends a all boundary values. This step can also be implemented for the three basic elliptic PDEs defined in the interior of a convex polygonal. For example, by employing either the classical Green's representation formula, or by performing the spectral analysis of the associated Lax pair [16], we find the following novel integral representation:

$$u(z,\bar{z}) = \frac{1}{4i\pi} \sum_{j=1}^{n} \int_{l_j} e^{i\frac{k}{2}(\lambda z - \frac{\bar{z}}{\lambda})} \hat{u}_j(\lambda) \frac{d\lambda}{\lambda}, \quad z \in \Omega, \tag{6.5a}$$

when $\{\hat{u}_j\}_1^n$ are defined in (6.4) in terms of all boundary values and $\{l_j\}_1^n$ are the rays in the complex λ-plane oriented towards infinity and defined by

$$l_j = \{\lambda \in \mathbb{C} : \arg\lambda = -\arg(z_{j+1} - z_j), \quad j = 1, \cdots, n, \quad z_{n+1} = z_1\}. \tag{6.5b}$$

For simple domains, it is possible, to impement step 3 of the Fokas method: using the global relations and their invariant properties, it is possible to express all transforms in terms of the given boundary data, using only algebraic manipulations. This has led to the analytic solution of several BVPs for which the usual approaches apparently fail [16].

For more complicated domains, the global relations suggest the novel numerical technique for the determination of the unknown boundary values, discussed earlier.

Further development

The rigorous foundation of the new method for *linear forced* evolution PDEs in Sobolev spaces is presented in [4], [17]. These results actually lead to a new approach for proving well posedness for *nonlinear* IBVPs. The crucial ingredient of this approach is to use for the linear version of the given nonlinear PDE, the formulae obtained via the new method. Earlier authors have been able to prove well posedness for IBVPs using ideas similar to those used in the treatment of initial-value problems. In particular, one first obtains a solution formula for the linear IBVP with forcing and then uses this formula to derive appropriate linear estimates. Subsequently, one replaces the forcing in the linear formula by the nonlinearity and uses the linear estimates together with a contraction mapping argument to deduce well-posedness of the nonlinear IBVP.

It is often the case, however, that even the derivation of the linear solution formula is somewhat technical and unintuitive, not to mention the derivation of the relevant linear estimates. The main advantage of the new method is that it yields explicit formulae for forced linear evolution equations with arbitrary number of derivatives. Thus, it is not surprising that these "naturally emerging" linear formulae can be used to establish local well-posedness of nonlinear evolution IBVPs through a contraction mapping approach.

Anthony Ashton employing the new method has developed a remarkable formalism for the rigorous analysis of elliptic PDEs, see for example [18].

For recent results regarding the characterization of the Dirichlet to Neumann map for integrable nonlinear evolution PDEs, see for example [19]-[22].

Linear evolution PDEs with either non-separable or other complicated boundary conditions are analyzed in [23]–[27].

The new method can be extended to three dimensions, see for example [5], [28]–[29]

Reviews of the Fokas method for linear and for integrable nonlinear PDEs are presented in [30] and [31], respectively.

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