Hybrid Photonics Computing. Example Sheet 2.

Due March 6, 2020

Kuramoto networks

1. Synchronization in Kuramoto networks

Kuramoto proposed a mathematically tractable model to describe the phenomenology of synchronization. He recognized that the most suitable case for analytical treatment should be the mean field approach. He proposed an all-to-all purely sinusoidal coupling, and then the governing equations for each of the oscillators in the system are:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad (i = 1, ..., N)\ , \tag{1}$$

where the factor $1/N$ is incorporated to ensure a good behavior of the model in the thermodynamic limit, $N \to \infty$, ω_i stands for the natural frequency of oscillator i, and K is the coupling constant. The frequencies ω_i are distributed according to some function $g(\omega)$, that is usually assumed to be unimodal and symmetric about its mean frequency Ω. Admittedly, due to the rotational symmetry in the model, we can use a rotating frame and redefine $\omega_i \to \omega_i + \Omega$ for all i and set $\Omega = 0$, so that the ω_i's denote deviations from the mean frequency.

The collective dynamics of the whole population is measured by the *macroscopic* complex order parameter,

$$r(t)\mathrm{e}^{\mathrm{i}\phi(t)} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j(t)}\ , \tag{2}$$

where the modulus $0 \le r(t) \le 1$ measures the phase coherence of the population and $\phi(t)$ is the average phase. The values $r \simeq 1$ and $r \simeq 0$ (where $\simeq$ stands for fluctuations of size $O(N^{-1/2})$) describe the limits in which all oscillators are either phase locked or move incoherently, respectively.

(i) Show that

$$\dot{\theta}_i = \omega_i + Kr\sin(\phi - \theta_i) \qquad (i = 1, ..., N)\ . \tag{3}$$

Equation (3) states that each oscillator interacts with all the others only through the mean field quantities r and ϕ. The first quantity provides a positive feedback loop to the system's collective rhythm: as r increases because the population becomes more coherent, the coupling between the oscillators is further strengthened and more of them can be recruited to take part in the coherent pack. Moreover, Eq. (3) allows to calculate the critical coupling K_c and to characterize the order parameter $\lim_{t\to\infty} r_t(K) = r(K)$. Looking for steady solutions, one assumes that $r(t)$ and $\phi(t)$ are constant. Next, without loss of generality, we can set $\phi = 0$, which leads to the equations of motion

$$\dot{\theta}_i = \omega_i - Kr\sin(\theta_i) \qquad (i = 1, ..., N)\ . \tag{4}$$

The solutions of Eq. (4) reveal two different types of long-term behavior when the coupling is larger than the critical value, K_c. On the one hand, a group of oscillators for which $|\omega_i| \le Kr$ are phase-locked at frequency Ω in the original frame according to the equation $\omega_i = Kr\sin(\theta_i)$. On the other hand, the rest of the oscillators for which $|\omega_i| > Kr$ holds, are drifting around the circle, sometimes accelerating and sometimes rotating at lower frequencies. Demanding some conditions for the stationary distribution $g(\omega)$ of drifting oscillators with frequency ω_i and phases θ_i, a self-consistent equation for r can be derived as

$$r = Kr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\cos^2\theta\right)g(\omega)d\theta,$$

where $\omega = Kr\sin(\theta)$. This equation admits a non-trivial solution,

$$K_c = \frac{2}{\pi g(0)}. \tag{5}$$

beyond which $r > 0$. Equation (5) is the Kuramoto mean field expression for the critical coupling at the onset of synchronization. Moreover, near the onset, the order parameter, r, obeys the usual square-root scaling law for mean field models, namely,

$$r \sim (K - K_c)^\beta \tag{6}$$

with $\beta = 1/2$.

Classical Complex-Valued Matter Fields

2. Complex-Valued Scalar Field as a Canonical Variable

(i) In classical Hamiltonian mechanics, the equations of motion are generated with the Hamiltonian function $H(\{q_j, p_j\})$ by

$$\dot{q_j} = \frac{\partial H}{\partial p_j}, \quad \dot{p_j} = -\frac{\partial H}{\partial q_j},$$

where a set of pairs of conjugated variables, $\{q_j, p_j\}, j = 1, 2..$ referred to as generalised coordinates and momenta, respectively. Show that by introducing $a_j = \alpha q_j + i\beta p_j$, where α and β are certain complex numbers, the equations of motion can be combined into a single equation

$$i\lambda \dot{a_j} = \frac{\partial H}{\partial a_j^*},$$

where you should specify λ.

[*Note that a_j is known as a complex canonical variable and λ can always be scaled to be 1 by an appropriate choice of α and β.*]

(ii) In analogy with (i), if the classical complex-valued scalar field $\psi(\mathbf{r}, t)$ is a complex canonical variable, then the dynamics is described by

$$i\partial_t \psi(\mathbf{r}, t) = \frac{\partial H[\psi, \psi^*]}{\partial \psi^*(\mathbf{r}, t)}. \tag{7}$$

Let the field ψ depend on some parameter γ: $\psi \equiv \psi(\mathbf{r}, \gamma)$. Show that for any functional $F[\psi, \psi^*]$ the following formula takes place:

$$\frac{\partial F}{\partial \gamma} = \int \left(\frac{\delta F}{\delta \psi}\frac{\delta \psi}{\delta \gamma} + \frac{\delta F}{\delta \psi^*}\frac{\delta \psi^*}{\delta \gamma} \right) d\mathbf{r}.$$

Therefore, show that $H[\psi, \psi^*]$ is conserved by the equation of motion, Eq.(7).

(iii) Assume that $H[\psi, \psi^*]$ in Eq. (7) has global $U(1)$ symmetry (H is invariant with respect to the transformation $\psi \to \psi \exp[i\phi]$, where ϕ does not depend on $\mathbf{r}$) and show that in this case the 'amount of the field' , $N = \int |\psi|^2 \, d\mathbf{r}$ is conserved.

[*Hint: use $\gamma = t$ in (ii) and $\gamma = \phi$ in (iii).*]

Equilibrium condensates and Adiabatic Annealing

3. Near-adiabatic ramps in interacting many-particle systems. Many experimental protocols for quantum annealing involve adiabatically changing a parameter in order to reach a desired quantum state. We consider ramps of the interaction from an initial value U_i to a final value U_f, occurring in time scale τ: $U(t) = U_i + \theta(t)(U_f - U_i)s(t/\tau)$, $s(0) = 0, s(\infty) = 1$. We focus on large but finite τ (near-adiabatic ramps) and two systems: Bose and Fermi condensates.

(i) **Bose condensate**. The BEC in a parabolic trapping potential, $V_{\text{ext}} = r^2/2$ in polar coordinates, is described by the dimensionless Gross-Pitaevskii equation (GPE)

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\nabla^2 \psi + [\frac{1}{2}r^2 + U(t)|\psi|^2]\psi, \tag{8}$$

where ψ is normalised, $\int_D |\psi|^2 \, d\mathbf{r} = 1$, so $U(t)$ is the effective interaction strength that contains a factor of the boson number N.

(ii) **Weakly interacting two-component fermions**. The continuum fermionic system (in three dimensions) often is treated using a hydrodynamic description. There are several similar formulations; we choose the so-called "time-dependent DFT", where the fermionic gas is described by a nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\nabla^2\psi + [\frac{1}{2}r^2 + \frac{1}{2}U(t)|\psi|^2 + \alpha|\psi|^{4/3}]\psi, \tag{9}$$

where $\alpha = \frac{1}{2}(3\pi^2)^{2/3}$. Both a paired superfluid ($U < 0$) and a Fermi liquid ($U > 0$) are described by the same formalism.

Assuming a Gaussian variational ansatz in 1D

$$\psi(x,t) = \frac{1}{[\sqrt{\pi}\sigma(t)]^{1/2}} \exp\left[-\frac{x^2}{2[\sigma(t)]^2} - i\beta(t)x^2\right] \tag{10}$$

in (i) and

$$\psi(x,t) = \frac{\sqrt{N}}{\sqrt{\pi^{3/2}\sigma(t)^3}} \exp\left[-\frac{r^2}{2[\sigma(t)]^2} - i\beta(t)r^2\right] \tag{11}$$

in (ii) in 3D spherical geometry together with the GPE Lagrangian

$$L = \frac{i}{2}\int_D \left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) - E[\psi], \tag{12}$$

where $E[\psi]$ is the functional ($i\psi_t = \partial E/\partial\psi^*$), formulate the Euler-Lagrange equations of motion and, therefore, give the evolution equations for the variational parameters $\sigma(t)$ and $\beta(t)$. Noting that the two parameters σ and β are not independent, eliminate β and obtain the equation of motion for the radius of the condensate $\sigma(t)$.

[*Note that the deviations from adiabaticity can be studied through the heating Q, which is the final energy at time t $\gg \tau$ minus the ground-state energy of the final Hamiltonian. This quantity is also called the residual energy or excess excitation energy, and may be thought of as the 'friction' due to imperfect adiabaticity.*]

4. Simulator: optical lattices of ultra-cold BECs

The full Bose condensate dynamics satisfies the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi + [V_{ext} + g_0 \mid \Phi \mid^2]\Phi, \tag{13}$$

where V_{ext} is the external potential given by the laser field

$$V_{ext}(\vec{r}) = U_L(x,y)\sin^2[2\pi z/\lambda] \tag{14}$$

where λ is the wavelength of the lasers (the spacing in the lattice is $\lambda/2$) and $U_L(x,y)$ is determined by the transverse intensity profile of the (nearly Gaussian) laser beams.

In the tight-binding approximation the condensate order parameter can be written as:

$$\Phi(\vec{r},t) = \sqrt{N_T}\sum_n \psi_n(t)\ \phi(\vec{r}-\vec{r}_n), \tag{15}$$

where N_T is the total number of atoms and $\phi(\vec{r}-\vec{r}_n)$ is the condensate wave function localized in the trap n with $\int d\vec{r}\ \phi_n\ \phi_{n+1} \simeq 0$, and $\int d\vec{r}\ \phi_n^2 = 1$. $\psi_n = \sqrt{\rho_n(t)}\,e^{i\theta_n(t)}$ is the n-th amplitude ($\rho_n = N_n/N_T$, where N_n and θ_n are the number of particles and phases in the trap n).

(i) Show that under these assumptions GPE reduces to the following equation:

$$i\frac{\partial\psi_n}{\partial t} = -\frac{1}{2}(\psi_{n-1} + \psi_{n+1}) + (\epsilon_n + \Lambda \mid \psi_n \mid^2)\psi_n \tag{16}$$

where you need to specify ϵ, Λ, K and state how you rescaled the time.

(ii) Show that Eq. (16) is the equation of motion $\dot{\psi}_n = \frac{\partial\mathcal{H}}{\partial(i\psi_n^*)}$, with $i\psi_n^*, \psi_n$ canonically conjugate variables, where you need to specify the Hamiltonian function $\mathcal{H}$.

(iii) Under which assumptions the Hamiltonian function $\mathcal{H}$ reduces to the XY Hamiltonian?