Hybrid Photonics Computing. Example Sheet 3.

Due May 10, 2020

Equilibrium quantum fluids

1. Simulator: optical lattices

The full Bose condensate dynamics satisfies the Gross-Pitaevskii equation

$$i\hbar\frac{\partial\Phi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Phi + [V_{ext} + g_0 \mid \Phi \mid^2]\Phi \tag{1}$$

where V_{ext} is the external potential given by the laser field

$$V_{ext}(\vec{r}) = U_L(x,y)\sin^2[2\pi z/\lambda] \tag{2}$$

where λ is the wavelength of the lasers (the spacing in the lattice is $\lambda/2$) and $U_L(x,y)$ is determined by the transverse intensity profile of the (nearly Gaussian) laser beams.

In the tight-binding approximation the condensate order parameter can be written as:

$$\Phi(\vec{r},t) = \sqrt{N_T}\sum_n \psi_n(t)\ \phi(\vec{r}-\vec{r}_n), \tag{3}$$

where N_T is the total number of atoms and $\phi(\vec{r}-\vec{r}_n)$ is the condensate wave function localized in the trap n with $\int d\vec{r}\ \phi_n\ \phi_{n+1} \simeq 0$, and $\int d\vec{r}\ \phi_n^2 = 1$. $\psi_n = \sqrt{\rho_n(t)}\,e^{i\theta_n(t)}$ is the n-th amplitude ($\rho_n = N_n/N_T$, where N_n and θ_n are the number of particles and phases in the trap n).

(i) Show that under these assumptions GPE reduces to the following equation:

$$i\frac{\partial\psi_n}{\partial t} = -\frac{1}{2}(\psi_{n-1}+\psi_{n+1}) + (\epsilon_n + \Lambda \mid \psi_n \mid^2)\psi_n \tag{4}$$

where you need to specify ϵ, Λ, K and state how you rescaled the time.

(ii) Show that Eq. (4) is the equation of motion $\dot{\psi}_n = \frac{\partial\mathcal{H}}{\partial(i\psi_n^*)}$, with $i\psi_n^*, \psi_n$ canonically conjugate variables, where you need to specify the Hamiltonian function $\mathcal{H}$.

(iii) Under which assumptions the Hamiltonian function $\mathcal{H}$ reduces to the XY Hamiltonian?

Kuramoto networks

2. Synchronization in Kuramoto networks

Kuramoto proposed a mathematically tractable model to describe the phenomenology of synchronization. He recognized that the most suitable case for analytical treatment should be the mean field approach. He proposed an all-to-all purely sinusoidal coupling, and then the governing equations for each of the oscillators in the system are:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\theta_i) \qquad (i=1,...,N)\ , \tag{5}$$

where the factor $1/N$ is incorporated to ensure a good behavior of the model in the thermodynamic limit, $N\to\infty$, ω_i stands for the natural frequency of oscillator i, and K is the coupling constant. The frequencies ω_i are distributed according to some function $g(\omega)$, that is usually assumed to be unimodal and symmetric about its mean frequency

Ω. Admittedly, due to the rotational symmetry in the model, we can use a rotating frame and redefine $\omega_i \to \omega_i + \Omega$ for all i and set $\Omega = 0$, so that the ω_i's denote deviations from the mean frequency.

The collective dynamics of the whole population is measured by the *macroscopic* complex order parameter,

$$r(t)e^{i\phi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j(t)} \ , \tag{6}$$

where the modulus $0 \leq r(t) \leq 1$ measures the phase coherence of the population and $\phi(t)$ is the average phase. The values $r \simeq 1$ and $r \simeq 0$ (where $\simeq$ stands for fluctuations of size $O(N^{-1/2})$) describe the limits in which all oscillators are either phase locked or move incoherently, respectively.

(i) Show that

$$\dot{\theta}_i = \omega_i + Kr\sin(\phi - \theta_i) \qquad (i = 1, ..., N) \ . \tag{7}$$

Equation (7) states that each oscillator interacts with all the others only through the mean field quantities r and ϕ. The first quantity provides a positive feedback loop to the system's collective rhythm: as r increases because the population becomes more coherent, the coupling between the oscillators is further strengthened and more of them can be recruited to take part in the coherent pack. Moreover, Eq. (7) allows to calculate the critical coupling K_c and to characterize the order parameter $\lim_{t\to\infty} r_t(K) = r(K)$. Looking for steady solutions, one assumes that $r(t)$ and $\phi(t)$ are constant. Next, without loss of generality, we can set $\phi = 0$, which leads to the equations of motion

$$\dot{\theta}_i = \omega_i - Kr\sin(\theta_i) \qquad (i = 1, ..., N) \ . \tag{8}$$

The solutions of Eq. (8) reveal two different types of long-term behavior when the coupling is larger than the critical value, K_c. On the one hand, a group of oscillators for which $|\omega_i| \leq Kr$ are phase-locked at frequency Ω in the original frame according to the equation $\omega_i = Kr\sin(\theta_i)$. On the other hand, the rest of the oscillators for which $|\omega_i| > Kr$ holds, are drifting around the circle, sometimes accelerating and sometimes rotating at lower frequencies. Demanding some conditions for the stationary distribution $g(\omega)$ of drifting oscillators with frequency ω_i and phases θ_i, a self-consistent equation for r can be derived as

$$r = Kr\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\cos^2\theta\right) g(\omega) d\theta,$$

where $\omega = Kr\sin(\theta)$. This equation admits a non-trivial solution,

$$K_c = \frac{2}{\pi g(0)}. \tag{9}$$

beyond which $r > 0$. Equation (9) is the Kuramoto mean field expression for the critical coupling at the onset of synchronization. Moreover, near the onset, the order parameter, r, obeys the usual square-root scaling law for mean field models, namely,

$$r \sim (K - K_c)^{\beta} \tag{10}$$

with $\beta = 1/2$.

Nonequilibrium quantum fluids

3. Coupling strength in a polariton dyad. Consider two unequally pumped polariton condensates with Gaussian pumping profiles. For two spatially separated condensates, we approximate the wave-function of the system as the sum of the two wavefunctions of the individually created condensates:

$$\psi(\mathbf{r}) \approx \Psi_1(|\mathbf{r} - \mathbf{r_1}|) + \Psi_2(|\mathbf{r} - \mathbf{r_2}|), \tag{11}$$

where the wavefunction of a condensate located at $\mathbf{r} = \mathbf{r_i}$ can be approximated by

$$\Psi_i(|\mathbf{r} - \mathbf{r_i}|) \approx \sqrt{\rho_i(|\mathbf{r} - \mathbf{r_i}|)}\exp[ik_{ci}|\mathbf{r} - \mathbf{r_i}| + i\theta_i], \tag{12}$$

where θ_i is the space independent part of the phase, k_{ci} is the maximum wave-vector $k(\mathbf{r})$ that polaritons reach within their lifetime by converting their potential to kinetic energy, and $\rho_i(|\mathbf{r} - \mathbf{r}_i|)$ is the density of the isolated condensate created by a single pumping source centered at $\mathbf{r}_i$.

The total number of polaritons across the dyad is given by $\mathcal{N} = \int |\psi(\mathbf{r})|^2 d\mathbf{r}$, where integration is over the entire area of the microcavity and

$$\mathcal{N} \approx \int |\Psi_1(|\mathbf{r} - \mathbf{r_1}|) + \Psi_2(|\mathbf{r} - \mathbf{r_2}|)|^2 d\mathbf{r} \tag{13}$$

where $\mathcal{N}_i = \int |\Psi_i(|\mathbf{r} - \mathbf{r}_i|)|^2 \, d\mathbf{r}$ is the number of polaritons of an individual condensate indexed by i.From all the possible phase differences between two polariton condensates, $\Delta\theta = \theta_1 - \theta_2 = [0, 2\pi)$, the one that maximises the number of particles $\mathcal{N}$ will condense first.

(1) Show that in the generic case of a polariton dyad with unequal populations, the system will reach threshold at the phase difference configuration $\Delta\theta$ that minimises

$$H_T = -(J \cos \Delta\theta + D \sin \Delta\theta), \tag{14}$$

where you need to identify J and D. H_T takes form of the sum of the symmetric Heisenberg exchange and the asymmetric Dzyaloshinskii-Moriya interactions, that are usually studied in the context of a contribution to the total magnetic exchange interaction between two neighboring magnetic spins.

(2) Show that for equally pumped condensates $D = 0$.

(3) Obtain analytical expressions of the coupling strengths J for equally pumped condensates by positioning the condensates at $\mathbf{r}_1 = (-d/2, 0)$ and $\mathbf{r}_2 = (d/2, 0)$, where $d = |\mathbf{r_1} - \mathbf{r_2}|$ is the separation distance, and transforming into elliptic coordinates (μ, ν) with

$$x = \frac{d}{2} \cosh \mu \cos \nu, \tag{15}$$

$$y = \frac{d}{2} \sinh \mu \sin \nu, \tag{16}$$

$$d^2\mathbf{r} = \frac{d^2}{4} (\sinh^2 \mu + \sin^2 \nu) d\mu d\nu, \tag{17}$$

where μ is a positive real number and $\nu \in [0, 2\pi)$. Assume an exponential decay of the amplitude for an individual condensate $\sqrt{\rho(|\mathbf{r} - \mathbf{r_i}|)} = A \exp(-\beta|\mathbf{r} - \mathbf{r_i}|)$, where A and β correlate with the shape of the pumping profile and show that

$$J = \pi A^2 d \left[\frac{1}{\beta} J_0(k_c d) K_1(\beta d) + \frac{1}{k_c} J_1(k_c d) K_0(\beta d) \right]. \tag{18}$$

(4) Comment on the behaviour of J if the pumping width is large (β is small).

4. Polariton simulators: taking into account light polarization. Consider the full system of equations describing the the left- and right-circular polariton states ψ_+ and ψ_+ in exciton-polariton condensates coupled to hot exciton reservoirs $N_\pm$:

$$\begin{aligned} i\partial_t \psi_\pm &= \Big(-(1 - i\eta_s N_+ - \eta_0 N_-)\nabla^2 + g_s N_+ + g_0 N_- + \alpha_s |\psi_\pm|^2 \\ &+ \alpha_0 |\psi_\mp|^2 \pm \frac{\Delta}{2} + \frac{i}{2}(R_s N_+ + R_0 N_- - \gamma_C) \Big) \psi_\pm + J\psi_\mp, \quad (19) \\ \partial_t N_\pm &= \tilde{P}_\pm - \Gamma_x N_\pm - (R_s |\psi_\pm|^2 + R_0 |\psi_\mp|^2) + \Gamma_s (N_\mp - N_\pm), \quad (20) \end{aligned}$$

where $N_\pm$ represent the densities of the incoherent reservoirs of spin-up and spin-down excitons providing gain to the condensate through stimulated bosonic scattering, γ_C characterizes the linear losses in the condensate, the blueshift of the polariton energy levels through Coulomb interaction is given by g_s and g_0 for the same- and cross-spin interaction strengths, $\alpha_{1,2}$ are the same- and cross-spin polariton interaction parameters, $R_{s,0}$ are the same- and opposite-spin gain from the two spin-polarized reservoirs to the condensate, $P_\pm$ represent the non-resonant pump intensities, Γ_x is the exciton lifetime, and Γ_s denotes the spin relaxation rate in the reservoir, Δ and J have the same meaning as in the previous Problem, $\eta_{s,0}$ describe the condensate energy relaxation due to the scattering with reservoirs.

Consider 1D chain of N polariton condensates created with N Gaussian beams $\tilde{P}_\pm = P_\pm \sum_{i=1}^N \exp[-\alpha(x - x_i)^2]$, where α characterises the inverse width of each pump centered at $x = x_i$. Use the tight-binding approximation $\psi_\pm = \sum_{i=1}^N a_i^\pm(t)\phi_i^\pm(x)$, $N_\pm = \sum_{i=1}^N k_i^\pm(t) n_i^\pm(x)$, $\phi_i^\pm = \phi^\pm(x - x_i)$, $n_i^\pm = n^\pm(x - x_i)$, and the assumptions used in class and in Problem 1 derive the rate equations on $a_i^\pm$, $k_i^\pm$. Clearly state your assumptions.

Comment on the use of the resulting networks for solving optimization problems.