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1. Construct a non-zero element of  $\mathcal{D}(\mathbf{R})$  that vanishes outside  $(0, 1)$ . Construct a non-zero element of  $\mathcal{D}(\mathbf{R}^n)$  that vanishes outside the ball  $B_\epsilon = \{x \in \mathbf{R}^n : |x| < \epsilon\}$ .
2. Given  $\varphi \in \mathcal{D}(X)$ , Taylor's theorem gives

$$\varphi(x+h) = \sum_{|\alpha| \leq N} \frac{h^\alpha}{\alpha!} \partial^\alpha \varphi(x) + R_N(x, h).$$

Prove that  $\text{supp}(R_N)$  is contained in some fixed compact  $K \subset X$  for  $|h|$  sufficiently small. Show also that  $\partial^\alpha R_N = o(|h|^N)$  uniformly in  $x$  for each multi-index  $\alpha$ , i.e. prove

$$\lim_{|h| \rightarrow 0} \left[ \frac{\sup_x |\partial^\alpha R_N(x, h)|}{|h|^N} \right] = 0$$

for each multi-index  $\alpha$ .

[Hint: you may find it convenient to use the following form of remainder

$$R_N(x, h) = \sum_{|\alpha|=N+1} \frac{h^\alpha}{\alpha!} (N+1) \int_0^1 (1-t)^N (\partial^\alpha \varphi)(x+th) dt,$$

and note that  $(N+1)! \sum_{|\alpha|=N+1} h^\alpha / \alpha! = (h_1 + \dots + h_n)^{N+1}$ .]

3. Which elements of  $\mathcal{D}(X)$  can be represented as a power series on  $X$ ?
4. Prove the  $C^\infty$  Urysohn lemma: if  $K$  is a compact subset of  $X \subset \mathbf{R}^n$ , show that one can find a  $\varphi \in \mathcal{D}(X)$  such that  $0 \leq \varphi \leq 1$  and  $\varphi = 1$  on a neighbourhood of  $K$ .  
 [Hint: Let  $\chi$  be a characteristic (indicator) function on a neighbourhood of  $K$  and smooth it off by taking the convolution with a test function that approximates  $\delta_0$ .]

5. Given  $T \in \mathcal{D}'(X)$ , the derivative  $\partial^\alpha T$  is defined by

$$\langle \partial^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \varphi \rangle \quad \forall \varphi \in \mathcal{D}(X).$$

Show that that  $\partial^\alpha T \in \mathcal{D}'(X)$ . If  $\text{ord}(T) = m$  what can you say about  $\text{ord}(\partial^\alpha T)$ ?

6. Let  $T \in \mathcal{D}'(X)$  and  $f \in C^\infty(X)$ . Prove that for each multi-index  $\alpha$

$$\partial^\alpha (Tf) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^\beta f \partial^{\alpha-\beta} T$$

in  $\mathcal{D}'(X)$  (take the case  $X = \mathbf{R}$  if you're still getting used to multi-indices).

7. Let  $\{x_k\}_{k \geq 1}$  be a sequence in  $X$  with no limit point in  $X$ . Consider the family of linear maps  $u_\alpha : \mathcal{D}(X) \rightarrow \mathbf{C}$  defined by

$$\langle u_\alpha, \varphi \rangle = \sum_{k=1}^{\infty} \partial^\alpha \varphi(x_k)$$

for each multi-index  $\alpha$ . For which  $\alpha$  is  $u_\alpha \in \mathcal{D}'(X)$ ? What is  $\text{ord}(u_\alpha)$ ?

8. Find the most general solution to the equations

$$(a) \quad u' = 1, \quad (b) \quad xu' = \delta_0, \quad (c) \quad (e^{2\pi i x} - 1) u' = 0$$

in  $\mathcal{D}'(\mathbf{R})$ .

9. Define the distribution  $u \in \mathcal{D}'(\mathbf{R}^2)$  by the locally integrable function

$$u(x, y) = \begin{cases} 1, & x \geq y \\ 0, & \text{otherwise.} \end{cases}$$

Show that  $\partial_x^2 u - \partial_y^2 u = 0$  in  $\mathcal{D}'(\mathbf{R}^2)$ . Can you give a physical interpretation of this result?

10. Compute  $\Delta(|x|^{2-n})$  in  $\mathcal{D}'(\mathbf{R}^n)$  for  $n \geq 3$ , i.e. compute

$$\langle \Delta(|x|^{2-n}), \varphi \rangle = \langle |x|^{2-n}, \Delta \varphi \rangle = \int \frac{\Delta \varphi}{|x|^{n-2}} dx$$

for arbitrary  $\varphi \in \mathcal{D}(\mathbf{R}^n)$ . Note that  $|x| = (x_1^2 + \dots + x_n^2)^{1/2}$  and  $\Delta = \sum_i (\partial/\partial x_i)^2$ .

[Hint: use  $\int dx = \int_{|x| \leq \epsilon} dx + \int_{|x| > \epsilon} dx$  and treat each integral separately.]

11. Let  $\{f_k\}_{k \geq 1}$  be the sequence of smooth functions defined by

$$f_k(x) = \frac{1}{\pi} \frac{k}{(kx)^2 + 1}.$$

Prove that  $f_k \rightarrow \delta_0$  in  $\mathcal{D}'(\mathbf{R})$ . Compute the limits

$$(a) \lim_{k \rightarrow \infty} k^2 x e^{-k^2 x^2}, \quad (b) \lim_{k \rightarrow \infty} k^3 e^{ikx}, \quad (c) \lim_{k \rightarrow \infty} \frac{\sin(kx)}{\pi x}$$

in  $\mathcal{D}'(\mathbf{R})$ .

12. Compute the limit

$$\lim_{k \rightarrow \infty} \left[ \frac{1}{2} + \sum_{m=1}^k \cos(\pi m x) \right]$$

in  $\mathcal{D}'(-1, 1)$ .

13. We define the *principal value* of  $1/x$ , written  $\text{p.v.}(1/x)$ , by

$$\left\langle \text{p.v.} \left( \frac{1}{x} \right), \varphi \right\rangle = \lim_{\epsilon \rightarrow 0} \int_{|x| > \epsilon} \frac{\varphi(x)}{x} dx$$

for all  $\varphi \in \mathcal{D}(\mathbf{R})$ . Prove that  $\text{p.v.}(1/x) \in \mathcal{D}'(\mathbf{R})$  and  $\text{ord}(\text{p.v.}(1/x)) = 1$ . Show that

$$\lim_{\epsilon \rightarrow 0} \left( \frac{1}{x - i\epsilon} \right) = \text{p.v.} \left( \frac{1}{x} \right) + i\pi \delta_0 \quad \text{in } \mathcal{D}'(\mathbf{R}).$$

14. Show that  $(\log|x|)' = \text{p.v.}(1/x)$  in  $\mathcal{D}'(\mathbf{R})$ .

15. Let  $f = f(z)$  be complex analytic on  $X \subset \mathbf{C} \simeq \mathbf{R}^2$ . Suppose  $f$  has zeros at  $\{z_i\}$  in  $X$  with multiplicities  $\{m_i\}$ . Prove that

$$\Delta \log |f| = 2\pi \sum_i m_i \delta_{z_i}$$

in  $\mathcal{D}'(X)$ , where  $\Delta = \partial_x^2 + \partial_y^2$  is the Laplacian on  $\mathbf{R}^2$ .

16. Define the distribution  $u \in \mathcal{E}'(\mathbf{R}^3)$  the locally integrable function

$$u(x) = \begin{cases} 1, & |x| \leq 1, \\ 0, & |x| > 1. \end{cases}$$

Prove that  $-\sum_i x_i (\partial u / \partial x_i) = d\sigma_2$  in  $\mathcal{E}'(\mathbf{R}^3)$ , where  $d\sigma_2$  is the surface element on the sphere  $S^2 \subset \mathbf{R}^3$ .