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1. Let  $u, v \in \mathcal{D}'(\mathbf{R}^n)$ , one of which has compact support. Show that the convolution  $u * v$ , as defined in your notes, is uniquely defined and gives rise to an element of  $\mathcal{D}'(\mathbf{R}^n)$ .
2. Show that if  $u, v, w \in \mathcal{D}'(\mathbf{R}^n)$  and at least two of them have compact support, then the convolution is associative, i.e.  $(u * v) * w = u * (v * w)$ .
3. Let  $\varphi \in \mathcal{D}(\mathbf{R})$  and choose  $\epsilon > 0$  sufficiently small so that  $\text{supp}(\varphi) \subset I_\epsilon$ , where  $I_\epsilon = (-1/\epsilon, 1/\epsilon)$ . Given that  $\varphi$  has a uniformly convergent Fourier series on  $I_\epsilon$  in the form

$$\varphi(x) = \sum_{n \in \mathbf{Z}} c_n e^{i\epsilon\pi n x}, \quad c_n = \frac{\epsilon}{2} \int \varphi(x) e^{-i\epsilon\pi n x} dx,$$

prove the Fourier inversion theorem on  $\mathcal{D}(\mathbf{R})$  by taking a suitable limit.

4. For  $\varphi \in \mathcal{S}(\mathbf{R}^n)$  prove that  $\sum_m \varphi(m) = \sum_n \hat{\varphi}(2\pi n)$ . This is the famous Poisson summation formula.
5. If  $u \in H^s(\mathbf{R}^n)$  show that  $D^\alpha u \in H^{s-|\alpha|}(\mathbf{R}^n)$ . If  $s > t$  show that  $H^s(\mathbf{R}^n) \subset H^t(\mathbf{R}^n)$ .
6. Show that  $\mathcal{S}(\mathbf{R}^n)$  is dense in  $L^2(\mathbf{R}^n) = H^0(\mathbf{R}^n)$  and deduce that  $\mathcal{S}(\mathbf{R}^n)$  is dense in  $H^s(\mathbf{R}^n)$ , i.e. prove that for each  $u \in H^s(\mathbf{R}^n)$  there is a sequence  $\{\varphi_m\}_{m \geq 1}$  in  $\mathcal{S}(\mathbf{R}^n)$  such that

$$\lim_{m \rightarrow \infty} \|u - \varphi_m\|_{H^s} = 0.$$

[Hint: Use Parseval's theorem.]

7. Prove that multiplication by a Schwartz function gives rise to a continuous map from  $H^s(\mathbf{R}^n)$  to itself, i.e.  $\|\varphi u\|_{H^s} \lesssim \|u\|_{H^s}$  for  $\varphi \in \mathcal{S}(\mathbf{R}^n)$ . You may assume Peetre's inequality: for  $\lambda, \mu \in \mathbf{R}^n$  and  $s \in \mathbf{R}$

$$\left( \frac{1 + |\lambda|^2}{1 + |\mu|^2} \right)^s \leq 2^{|s|} (1 + |\lambda - \mu|^2)^{|s|}.$$

8. For  $u \in \mathcal{S}'(\mathbf{R}^n)$  show that  $(D^\alpha u)^\wedge = \lambda^\alpha \hat{u}$  and  $(x^\beta u)^\wedge = (-D)^\beta \hat{u}$  for all multi-indices  $\alpha, \beta$ .
9. For  $u \in \mathcal{E}'(\mathbf{R}^n)$  show that  $\hat{u}(\lambda) = \langle u(x), e^{-i\lambda \cdot x} \rangle$ . Deduce that if  $u \in \mathcal{E}'(\mathbf{R}^n)$  there exists some  $t \in \mathbf{R}$  such that  $u \in H^t(\mathbf{R}^n)$ .
10. Suppose that  $u_1, u_2 \in \mathcal{S}'(\mathbf{R}^n)$  and at least one of  $u_1$  and  $u_2$  has compact support. Show that their convolution  $u_1 * u_2 \in \mathcal{S}'(\mathbf{R}^n)$  and  $(u_1 * u_2)^\wedge = \hat{u}_1 \hat{u}_2$ .
11. Show that  $e^{-\epsilon x} H \rightarrow H$  in  $\mathcal{S}'(\mathbf{R})$  as  $\epsilon \rightarrow 0$ . Hence show

$$\hat{H} = \pi \delta_0 - \text{ip.v.} \left( \frac{1}{x} \right)$$

in  $\mathcal{S}'(\mathbf{R})$ .

12. The Riemann-Lebesgue lemma states that if  $u \in L^1(\mathbf{R})$  then  $|\hat{u}(\lambda)| \rightarrow 0$  as  $|\lambda| \rightarrow \infty$ . Prove this result by considering the substitution  $x = x' + \pi/\lambda$  in the integral defining  $\hat{u}(\lambda)$ .
13. If  $u \in H^m(\mathbf{R}^n)$  with  $m \in \mathbf{N}$ , use Parseval's theorem to show that

$$\sum_{|\alpha| \leq m} \int |D^\alpha u|^2 dx < \infty.$$

Prove the converse.

14. Let  $\mathcal{O}(\mathbf{R}^n)$  denote the space of smooth functions that grow no faster than a polynomial. Show that  $\mathcal{O}(\mathbf{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$ . Fix  $\varphi \in \mathcal{S}(\mathbf{R}^n)$  with  $\varphi(0) = 1$ . For  $u \in \mathcal{O}(\mathbf{R}^n)$  we define

$$\hat{u}_\epsilon(\lambda) = \int e^{-i\lambda \cdot x} \varphi(\epsilon x) u(x) dx.$$

Show that  $\hat{u} = \lim_{\epsilon \rightarrow 0} \hat{u}_\epsilon$  in  $\mathcal{S}'(\mathbf{R}^n)$ .

15. Compute the Fourier transform in  $\mathcal{S}'(\mathbf{R})$  of the function

$$u(x) = \frac{x}{1+x^2}.$$

For which  $s \in \mathbf{R}$  is  $u \in H^s(\mathbf{R})$ ?

16. Prove that  $D^\alpha \delta_0 \in H^s(\mathbf{R}^n)$  if and only if  $s < -|\alpha| - \frac{1}{2}n$ .

17. Let  $\Gamma = \{x \in \mathbf{R}^n : x \cdot n = 0\}$  be a hyperplane with surface element  $d\sigma$  and normal  $n$ . Let  $\chi \in \mathcal{S}(\mathbf{R}^n)$  be a fixed Schwartz function and set  $d\mu = \chi d\sigma$ . This defines a distribution  $\mu_\Gamma \in \mathcal{S}'(\mathbf{R}^n)$  by

$$\langle \mu_\Gamma, \varphi \rangle = \int_\Gamma \varphi d\mu$$

for each  $\varphi \in \mathcal{S}(\mathbf{R}^n)$ . Prove that

$$\hat{\mu}_\Gamma(\lambda) = \int_\Gamma e^{-i\lambda \cdot x} d\mu(x).$$

Classify the large  $|\lambda|$  behaviour of  $\hat{\mu}_\Gamma$ . For which  $s \in \mathbf{R}$  is  $\mu_\Gamma \in H^s(\mathbf{R}^n)$ ?

18. Suppose that  $u \in \mathcal{S}'(\mathbf{R}^n)$  and  $\Delta^2 u + u \in H^s(\mathbf{R}^n)$ . Prove that  $u \in H^{s+4}(\mathbf{R}^n)$ .

19. Compute the Fourier transforms of the functions

$$(a) \operatorname{sgn}(x), \quad (b) \arctan(x), \quad (c) x \log|x| - x, \quad (d) \exp(i\omega x^2)$$

in  $\mathcal{S}'(\mathbf{R})$ , where  $\omega \in \mathbf{R}$ .

20. Show that the Fourier transform of  $(x_1 + ix_2)^{-1}$  is proportional to itself and find the constant of proportionality.

21. Suppose  $f \in C(\mathbf{R})$  and  $f(x) - 1/x = O(1/x^2)$  as  $|x| \rightarrow \infty$ . Prove that

$$\lim_{\epsilon \rightarrow 0} \left[ \hat{f}(-\epsilon) - \hat{f}(+\epsilon) \right] = 2\pi i.$$

22. If  $\varphi \in \mathcal{D}(\mathbf{R}^n)$  and  $\operatorname{supp}(\varphi) \subset B_\delta$  show that  $\hat{\varphi}(z)$  is an entire function and there exist constants  $C_m$  such that

$$|\hat{\varphi}(z)| \leq C_m (1 + |z|)^{-m} e^{\delta |\operatorname{Im} z|}$$

for  $m = 0, 1, 2, \dots$  and  $z \in \mathbf{C}^n$ . Prove the converse.