Classical and Quantum Solitons
Example Sheet 2

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1 Using $(a,b) = \frac{1}{2}(\overline{a}b + a\overline{b})$ check that $\partial_j(\Phi,\Psi) = (D_j\Phi,\Psi) + (\Phi, D_j\Psi)$ and $\partial_j(\Phi, D_j\Psi) = (D_j\Phi, D_j\Psi) + (\Phi, D_jD_j\Psi)$. Derive the Euler-Lagrange equations for the functional

$$V_\lambda(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2}\Big[B^2 + |D\Phi|^2 + \frac{\lambda}{4}(1-|\Phi|^2)^2\Big]d^2x, \qquad (1)$$

where $B = \partial_1 A_2 - \partial_2 A_1$ and $D_j\Phi = (\nabla - iA)_j\Phi$. Substitute the radial ansatz $\Phi = f_N(r)e^{iN\theta}$ and $A = N\alpha_N(r)d\theta$ into these Euler-Lagrange equations and hence obtain coupled ODEs satisfied by f_N, α_N.

2 Substitute the radial ansatz from the previous question into the energy functional (1) to obtain a radial energy functional. Calculate the corresponding Euler-Lagrange equations and check they are the same as those obtained in the previous question. The charge N radial vortex is a solution of this form with f_N, α_N having limits 1 as $r \to \infty$ and zero as $r \to 0$. Calculate the topological charge density $j^0 = \frac{1}{2}\epsilon_{ab}\epsilon_{ij}\partial_i\Phi^a\partial_j\Phi^b$ for this radial vortex, and verify that

$$N = \lim_{R\to+\infty}\frac{1}{2\pi}\int_{|x|=R}\langle i\Phi, d\Phi\rangle \;=\; \frac{1}{2\pi}\int_{\mathbb{R}^2} j^0\,dx\,. \qquad (2)$$

3 Consider the Ginzburg-Landau energy with $\lambda = 1$. Assuming that the degree/winding number (2) is a *negative* integer, carry out the Bogomolny argument with appropriate modification from the $N > 0$ case to deduce the Bogomolny bound $V_1 \geq \pi|N|$. Obtain the corresponding first order Bogomolny equations whose solvability provides configurations which saturate the Bogomolny bound.

4 Carry out the Bogomolny argument for the functional

$$W(A,\Phi) = \frac{1}{2}\int_{\mathbb{R}^2}\Big[\frac{1}{|\Phi|^2}B^2 + |D\Phi|^2 + \frac{|\Phi|^2}{4}(1-|\Phi|^2)^2\Big]d^2x, \qquad (3)$$

and obtain the first order Bogomolny equations whose solutions minimize W.

5 The Ginzburg-Landay energy functional on the hyperbolic disc $D = \{z : |z| < 1\} \subset \mathbb{C}$ is

$$\begin{aligned} V(A, \Phi) &= \frac{1}{2} \int_D \left[e^{2\rho} B^2 + |D\Phi|^2 + \frac{e^{2\rho}}{4} (1 - |\Phi|^2)^2 \right] d^2x, \\ &= \frac{1}{2} \int_D \left[e^{-2\rho} (\partial_x A_2 - \partial_y A_1)^2 + |D\Phi|^2 + \frac{e^{2\rho}}{4} (1 - |\Phi|^2)^2 \right] d^2x, \end{aligned}$$

where the metric is $g = e^{2\rho}(dx^2 + dy^2)$, $z = x + iy$, $B = e^{-2\rho}(\partial_x A_2 - \partial_y A_1)$ and $D\Phi = (\nabla - iA)\Phi$ as usual. Carry out the Bogomolny argument to obtain the first order Bogomolny equations for the minimizers.

6 In the previous question consider the case $e^{2\rho} = \frac{8}{(1-|z|^2)^2}$, and define ψ by

$$\psi = u - \ln(1 - z\bar{z}) + \ln 2$$

where $u = \ln |\Phi|$. Show that $\Delta\psi = e^{2\psi}$ (Liouville equation). Verify that this latter equation has solution

$$\psi = \frac{1}{2} \ln \frac{4|g'(z)|^2}{(1 - |g(z)|^2)^2}$$

for any holomorphic function $g : D \to D$, and hence obtain explicit radially symmetric vortex solutions.