Part III **Prof. N.S. Manton**
Easter Term 2017

Classical and Quantum Solitons
Examples 3 – Skyrmions

1. Starting from the Skyrme Lagrangian with $m = 0$,

$$L = \int \left\{ -\frac{1}{2}\mathrm{Tr}(R_\mu R^\mu) + \frac{1}{16}\mathrm{Tr}([R_\mu, R_\nu][R^\mu, R^\nu]) \right\} d^3x\,,$$

derive the field equation

$$\partial_\mu \left(R^\mu + \frac{1}{4}[R^\nu, [R_\nu, R^\mu]] \right) = 0\,.$$

Find the linearized field equation for the pion fields, when U is close to 1_2.

2. Write down the energy integral for static Skyrme fields, including the pion mass term. Find the form of Derrick's theorem satisfied by static Skyrmions.

3. Show that the 3-volume of a 3-sphere of unit radius is $2\pi^2$. Calculate the 4-volume of the 4-ball of unit radius.

4. Consider the map $U : S^3 \to SU(2)$ defined by $U(x_1, x_2, x_3, x_4) = x_4 + ix_1\tau_1 + ix_2\tau_2 + ix_3\tau_3$, where τ_1, τ_2, τ_3 are Pauli matrices and $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$. What is its degree? By calculating $\mathrm{Tr}(dU\,U^{-1} \wedge dU\,U^{-1} \wedge dU\,U^{-1})$ at the point on S^3 where $x_4 = 1$, or otherwise, deduce that the formula

$$\deg U = \frac{1}{24\pi^2} \int_{S^3} \mathrm{Tr}(dU\,U^{-1} \wedge dU\,U^{-1} \wedge dU\,U^{-1})$$

is correctly normalized.

5. Show that if $z = \tan\frac{1}{2}\theta\, e^{i\phi}$ then the corresponding unit vector on S^2 is

$$\hat{\mathbf{x}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) = \frac{1}{1+|z|^2}(\bar{z}+z, i(\bar{z}-z), 1-|z|^2)\,.$$

6. Let G denote the symmetry group SO(3) $\times$ SO(3) of rotations and isorotations of the Skyrme model. The $B = 2$ Skyrmion has a discrete end-over-end symmetry. After fixing the Skyrmion's orientation, find the combination $g \in G$ of rotation and isorotation that corresponds to this symmetry. Find a simple curve connecting the identity of G to g, and consider how the action of its group elements produces a closed loop of Skyrme field configurations (actually Skyrmions). Is this loop contractible or not? Hence write down the constraint on the quantum states Ψ of the Skyrmion associated with the symmetry g.

[Hint: Consider deforming the $B = 2$ Skyrmion into a chain of two separated $B = 1$ Skyrmions.]

7. Find the Möbius transformation for a 120° rotation about the $(1, 1, 1)$ axis in space. [Hint: Acting on z it sends $0 \to 1 \to i \to 0$.] Find the effect of this transformation on the rational

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map $R_4(z)$ with octahedral symmetry given in lectures. Hence find the corresponding constraint on the rotational/isorotational wavefunction of the $B = 4$ Skyrmion. Find the constraint corresponding to the 90° rotational symmetry about the x_3-axis. [Hint: The FR factors are +1 for both these – try to verify this.]

8. Determine the relations between baryon number B and the 3rd component of isospin I_3, and the proton and neutron numbers Z and N of a nucleus.

Estimate how the spin and isospin moments of inertia of a Skyrmion increase with baryon number B, for large B. Hence estimate the difference in energy betweeen the lowest spin 0 and spin 2 states of an even-even nucleus, and the asymmetry energy of a nucleus (the term proportional to $(Z - N)^2$).

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