3P1a **Quantum Field Theory: Example Sheet 1** Michaelmas 2016

Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk. *Starred questions may be handed in to your supervisor for feedback prior to the class* **by Part III maths students only**.

1. A string of length a, mass per unit length σ and under tension T is fixed at each end. The Lagrangian governing the time evolution of small transverse displacements $y(x,t)$ is

$$L = \int_0^a dx \left[\frac{\sigma}{2} \left(\frac{\partial y}{\partial t} \right)^2 - \frac{T}{2} \left(\frac{\partial y}{\partial x} \right)^2 \right]$$

where x identifies position along the string from one end point. By expressing the displacement as a sine series Fourier expansion of the form

$$y(x,t) = \sqrt{\frac{2}{a}} \sum_{n=1}^{\infty} q_n(t) \sin\left(\frac{n\pi x}{a} \right)$$

show that the Lagrangian becomes

$$L = \sum_{n=1}^{\infty} \left[\frac{\sigma}{2} \dot{q}_n^2 - \frac{T}{2} \left(\frac{n\pi}{a} \right)^2 q_n^2 \right] .$$

Derive the equations of motion. Hence show that the string is equivalent to an infinite set of decoupled harmonic oscillators with frequencies

$$\omega_n = \sqrt{\frac{T}{\sigma}} \left(\frac{n\pi}{a} \right) .$$

2. Show directly that if $\phi(x)$ satisfies the Klein-Gordon equation, then $\phi(\Lambda^{-1}x)$ also satisfies this equation for any Lorentz transformation Λ.

3. The motion of a complex field $\psi(x)$ is governed by the Lagrangian density

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \frac{\lambda}{2} (\psi^* \psi)^2.$$

Write down the Euler-Lagrange field equations for this system. Verify that the Lagrangian is invariant under the infinitesimal transformation

$$\delta\psi = i\alpha\psi, \qquad \delta\psi^* = -i\alpha\psi^*.$$

Derive the Noether current associated with this transformation and verify explicitly that it is conserved using the field equations satisfied by ψ.

4. Verify that the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2} m^2 \phi_a \phi_a$$

for a triplet of real fields ϕ_a, where $a \in \{1,2,3\}$ is invariant under the infinitesimal $SO(3)$ rotation by θ

$$\phi_a \to \phi_a + \theta \epsilon_{abc} \eta_b \phi_c$$

where η_a is a unit vector. Compute the Noether current j^μ. Deduce that the three quantities

$$Q_a = \int d^3x \ \epsilon_{abc} \dot{\phi}_b \phi_c$$

are all conserved and verify this directly using the field equations satisfied by ϕ_a.

5*. A Lorentz transformation $x^a \to x'^a = \Lambda^a{}_b x^b$ is such that it preserves the Minkowski metric η_{ab}, meaning that $\eta_{ab} x^a x^b = \eta_{ab} x'^a x'^b$ for all x. Show that this implies that

$$\eta_{ab} = \eta_{cd} \Lambda^c{}_a \Lambda^d{}_b \,.$$

Use this result to show that an infinitesimal transformation of the form

$$\Lambda^a{}_b = \delta^a{}_b + \omega^a{}_b$$

is a Lorentz transformation when ω^{ab} is antisymmetric: i.e. $\omega^{ab} = -\omega^{ba}$.

Write down the matrix form for $\omega^a{}_b$ that corresponds to a rotation through an infinitesimal angle θ about the x^3-axis. Do the same for a boost along the x^1-axis by an infinitesimal velocity v.

6*. Consider the infinitesimal form of the Lorentz transformation derived in the previous question: $x^\mu \to x^\mu + \omega^\mu_\nu x^\nu$. Show that a scalar field transforms as

$$\phi(x) \to \phi'(x) = \phi(x) - \omega^\mu{}_\nu x^\nu \partial_\mu \phi(x)$$

and hence show that the variation of the Lagrangian density is a total derivative

$$\delta \mathcal{L} = -\partial_\mu (\omega^\mu{}_\nu x^\nu \mathcal{L}).$$

Using Noether's theorem, deduce the existence of the conserved current

$$j^\mu = -\omega^\rho{}_\nu [T^\mu_\rho x^\nu].$$

The three conserved charges arising from spatial rotational invariance define the *total angular momentum* of the field. Show that these charges are given by

$$Q_i = \frac{1}{2} \epsilon_{ijk} \int d^3x \ \left(x^j T^{0k} - x^k T^{0j} \right).$$

Derive the conserved charges arising from invariance under Lorentz boosts. Show that they imply

$$\frac{d}{dt} \int d^3x \ \left(x^i T^{00} \right) = \text{constant}$$

and interpret this equation.

7. Maxwell's Lagrangian for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is the 4-vector potential. Show that $\mathcal{L}$ is invariant under gauge transformations

$$A_\mu \to A_\mu + \partial_\mu \xi,$$

where $\xi = \xi(x)$ is a scalar field with arbitrary (differentiable) dependence on x.

Using Noether's theorem, and the spacetime translational invariance of the action, to construct the energy momentum tensor $T^{\mu\nu}$ for the electromagnetic field. Show that the resulting object is neither symmetric nor gauge invariant. Consider a new tensor given by

$$\Theta^\mu = T^{\mu\nu} - F^{\rho\mu}\partial_\rho A^\nu.$$

Show that this object also defines four currents. Moreover, show that it is symmetric, gauge invariant and traceless.

8. The Lagrangian density for a massive vector field C_μ is given by

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 C_\mu C^\mu,$$

where $F_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$. Derive the equations of motion and show that when $m \neq 0$ they imply

$$\partial_\mu C^\mu = 0.$$

Further show that C_0 can be eliminated completely in terms of other fields by

$$\partial_i \partial^i C_0 + m^2 C_0 = \partial^i \dot{C}_i. \tag{1}$$

Construct the canonical momenta Π_i conjugate to C_i where $i \in \{1, 2, 3\}$ and show that the canonical momentum conjugate to C_0 is vanishing. Construct the Hamiltonian density $\mathcal{H}$ in terms of C_0, C_i and Π_i (NB: don't be concerned that the canonical momentum for C_0 is vanishing. C_0 is non-dynamical; it is determined entirely in terms of the other fields using Eq. (1)).

9. A class of interesting theories is invariant under the scaling of all lengths by

$$x^\mu \to (x')^\mu = \lambda x^\mu \text{ and } \phi(x) \to \phi'(x) = \lambda^{-D}\phi(\lambda^{-1}x). \tag{2}$$

Here, D is called the *scaling dimension* of the field. Consider the action for a real scalar field given by

$$S = \int d^4x \ \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - g\phi^p.$$

Find the scaling dimension D such that the derivative terms remain invariant. For what values of m and p is the scaling in Eq. (2) a symmetry of the theory? How do these conclusions change for a scalar field living in an $(n+1)-$dimensional spacetime instead of a $3+1$ dimensional spacetime?

In $3+1$ dimensions, use Noether's theorem to construct the conserved current D^μ associated with scaling invariance.