3P1c **Quantum Field Theory: Example Sheet 3** Michaelmas 2016

Corrections and suggestions should be emailed to B.C.Allanach@damtp.cam.ac.uk. *Starred questions may be handed in to your supervisor for feedback prior to the class.*

1. The Weyl representation of the Clifford algebra is

$$\gamma^0 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \quad , \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

Show that these indeed satisfy $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbf{1}$. Find a unitary matrix U such that $(\gamma')^a = U\gamma^a U^\dagger$, where $(\gamma')^a$ form the Dirac representation of the Clifford algebra

$$(\gamma')^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \quad , \quad (\gamma')^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} .$$

2. Show that if $\left\{\gamma^a,\, \gamma^b\right\} = 2\eta^{ab}$, then

$$\left[\gamma^a\gamma^b\,,\, \gamma^c\gamma^d\right] = 2\eta^{bc}\gamma^a\gamma^d - 2\eta^{ac}\gamma^b\gamma^d + 2\eta^{bd}\gamma^c\gamma^a - 2\eta^{ad}\gamma^c\gamma^b\,.$$

Show further that $S^{ab} \equiv \frac{1}{4}\left[\gamma^a\,,\, \gamma^b\right] = \frac{1}{2}(\gamma^a\gamma^b - \eta^{ab})$. Use this to confirm that the matrices S^{ab} form a representation of the Lie algebra of the Lorentz group.

3. Using just the algebra $\{\gamma^a,\ \gamma^b\} = 2\eta^{ab}$ (that is to say without resorting to any particular representation of the gamma matrices), and defining $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, $\not{p} = p_a\gamma^a$ and $S^{ab} \equiv \frac{1}{4}\left[\gamma^a,\ \gamma^b\right]$, prove the following results:

   (a) $\mathrm{Tr}\gamma^a = 0$

   (b) $\mathrm{Tr}(\gamma^a\gamma^b) = 4\eta^{ab}$

   (c) $\mathrm{Tr}(\gamma^a\gamma^b\gamma^c) = 0$

   (d) $(\gamma^5)^2 = 1$

   (e) $\mathrm{Tr}\gamma^5 = 0$

   (f) $\not{p}\not{q} = 2p\cdot q - \not{q}\not{p} = p\cdot q + 2S^{ab}p_a q_b$

   (g) $\mathrm{Tr}(\not{p}\not{q}) = 4p\cdot q$

   (h) $\mathrm{Tr}(\not{p}_1 \ldots \not{p}_n) = 0$ if n is odd

   (i) $\mathrm{Tr}(\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4\left[(p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - (p_1\cdot p_3)(p_2\cdot p_4)\right]$

   (j) $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2) = 0$

   (k) $\gamma_a\not{p}\gamma^a = -2\not{p}$

   (l) $\gamma_a\not{p}_1\not{p}_2\gamma^a = 4p_1\cdot p_2$

   (m) $\gamma_\mu\not{p}_1\not{p}_2\not{p}_3\gamma^\mu = -2\not{p}_3\not{p}_2\not{p}_1$

   (n) $\mathrm{Tr}(\gamma^5\not{p}_1\not{p}_2\not{p}_3\not{p}_4) = 4i\,\epsilon_{abcd}\,p_1^a\,p_2^b\,p_3^c\,p_4^d$

4*. The plane-wave solutions to the Dirac equation are

$$u^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix} \text{ and } v^s(\vec{p}) = \begin{pmatrix} \sqrt{p\cdot\sigma}\xi^s \\ -\sqrt{p\cdot\bar{\sigma}}\xi^s \end{pmatrix},$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$ and ξ^s, with $s \in \{1,2\}$, is a basis of orthonormal two-component spinors, satisfying $(\xi^r)^\dagger \cdot \xi^s = \delta^{rs}$. Show that

$$\begin{aligned} u^r(\vec{p})^\dagger \cdot u^s(\vec{p}) &= 2p_0\delta^{rs} \\ \bar{u}^r(\vec{p}) \cdot u^s(\vec{p}) &= 2m\delta^{rs} \end{aligned} \tag{1}$$

and similarly,

$$\begin{aligned} v^r(\vec{p})^\dagger \cdot v^s(\vec{p}) &= 2p_0\delta^{rs} \\ \bar{v}^r(\vec{p}) \cdot v^s(\vec{p}) &= -2m\delta^{rs}. \end{aligned} \tag{2}$$

Show also that the orthogonality condition between u and v is

$$\bar{u}^s(\vec{p}) \cdot v^r(\vec{p}) = 0,$$

while taking the inner product using † requires an extra minus sign

$$u^s(\vec{p})^\dagger \cdot v^r(-\vec{p}) = 0. \tag{3}$$

5. Using the same notation as Question 4, show that

$$\sum_{s=1}^{2} u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m, \tag{4}$$

$$\sum_{s=1}^{2} v^s(\vec{p})\bar{v}^s(\vec{p}) = \not{p} - m, \tag{5}$$

where, rather than being contracted, the two spinors on the left-hand side are placed back to back to form a 4×4 matrix.

6. The Fourier decomposition of the Dirac field operator $\psi(x)$ and the conjugate field $\psi^\dagger(\vec{x})$ is given by

$$\begin{aligned} \psi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left[b^s_{\vec{p}} u^s(\vec{p}) e^{i\vec{p}\cdot\vec{x}} + c^{s\dagger}_{\vec{p}} v^s(\vec{p}) e^{-i\vec{p}\cdot\vec{x}} \right], \\ \psi^\dagger(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \sum_{s=1}^{2} \left[b^{s\dagger}_{\vec{p}} u^s(\vec{p})^\dagger e^{-i\vec{p}\cdot\vec{x}} + c^s_{\vec{p}} v^s(\vec{p})^\dagger e^{i\vec{p}\cdot\vec{x}} \right]. \end{aligned} \tag{6}$$

The creation and annihilation operators are taken to satisfy

$$\begin{aligned} \{b^r_{\vec{p}}, b^{s\dagger}_{\vec{q}}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}), \\ \{c^r_{\vec{p}}, c^{s\dagger}_{\vec{q}}\} &= (2\pi)^3 \delta^{rs}\, \delta^{(3)}(\vec{p}-\vec{q}), \end{aligned}$$

with all other anticommutators vanishing. Show that these imply that the field and its conjugate field satisfy the anti-commutation relations

$$\begin{aligned} \{\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})\} &= \{\psi^\dagger_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} = 0, \\ \{\psi_\alpha(\vec{x}), \psi^\dagger_\beta(\vec{y})\} &= \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}). \end{aligned}$$

Note: the calculation is very similar to that for the bosonic field, but at some point you will need to make use of the identities Eqs. (4),(5).

7. Using the results of Question 6, show that the quantum Hamiltonian

$$H = \int d^3x\ \bar{\psi}(-i\gamma^i\partial_i + m)\psi$$

can be written, after normal ordering, as

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}} \sum_{s=1}^{2} \left[b_{\vec{p}}^{r\dagger} b_{\vec{p}}^{r} + c_{\vec{p}}^{r\dagger} c_{\vec{p}}^{r} \right].$$

Note: again, the calculation is very similar to that of the bosonic field. This time you will need to make use of the identities in Eqs. (1), (2) and (3).

8*. The purpose of this question is to give you a glimpse into the spin-statistics theorem. This theorem roughly says that if you try to quantize a field with the wrong statistics, bad things will happen. Here we'll see what goes wrong if you try to quantize a spin 1/2 Dirac field as a boson. We start with the usual decomposition given in Eq. 6. This time, we choose the creation and annihilation operators to satisfy bosonic commutation relations rather than fermionic anti-commutation ones:

$$\begin{aligned} [b_{\vec{p}}^{r}, b_{\vec{q}}^{s\dagger}] &= (2\pi)^3 \delta_{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \\ [c_{\vec{p}}^{r}, c_{\vec{q}}^{s\dagger}] &= -(2\pi)^3 \delta_{rs}\, \delta^{(3)}(\vec{p}-\vec{q}) \end{aligned}$$

with all other commutators vanishing. Note the strange minus sign for the c operators. Repeat the calculation of Question 6 to show that these are equivalent to the commutation relations

$$\begin{aligned} [\psi_\alpha(\vec{x}), \psi_\beta(\vec{y})] &= \left[\psi_\alpha^\dagger(\vec{x}), \psi_\beta^\dagger(\vec{y})\right] = 0, \\ \left[\psi_\alpha(\vec{x}), \psi_\beta^\dagger(\vec{y})\right] &= \delta_{\alpha\beta}\delta^{(3)}(\vec{x}-\vec{y}). \end{aligned}$$

Now repeat the calculation of Question 7, to show that, after normal ordering, the Hamiltonian is given by

$$H = \int \frac{d^3p}{(2\pi)^3}\, E_{\vec{p}} \sum_{r}^{2} \left[b_{\vec{p}}^{r\dagger} b_{\vec{p}}^{r} - c_{\vec{p}}^{r\dagger} c_{\vec{p}}^{r} \right].$$

This Hamiltonian is not bounded below: you can lower the energy indefinitely by creating more and more c particles. This is the reason a theory of bosonic spin 1/2 particles is sick.