

Symmetries, Fields and Particles. Examples 1.

1. $O(n)$ consists of $n \times n$ real matrices M satisfying $M^T M = I$. Check that $O(n)$ is a group. $U(n)$ consists of $n \times n$ complex matrices U satisfying $U^\dagger U = I$. Check similarly that $U(n)$ is a group.

Verify that $O(n)$ and $SO(n)$ are the subgroups of real matrices in, respectively, $U(n)$ and $SU(n)$. By considering how $U(n)$ matrices act on vectors in \mathbb{C}^n , and identifying \mathbb{C}^n with \mathbb{R}^{2n} , show that $U(n)$ is a subgroup of $SO(2n)$.

2. Show that for matrices $M \in O(n)$, the first column of M is an arbitrary unit vector, the second is a unit vector orthogonal to the first, ..., the k th column is a unit vector orthogonal to the span of the previous ones, etc. Deduce the dimension of $O(n)$. By similar reasoning, determine the dimension of $U(n)$.

Show that any column of a unitary matrix U is not in the (complex) linear span of the remaining columns.

3. Consider the real 3×3 matrix,

$$R(\mathbf{n}, \theta)_{ij} = \cos \theta \delta_{ij} + (1 - \cos \theta) n_i n_j - \sin \theta \epsilon_{ijk} n_k$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit vector in \mathbb{R}^3 . Verify that \mathbf{n} is an eigenvector of $R(\mathbf{n}, \theta)$ with eigenvalue one. Now choose an orthonormal basis for \mathbb{R}^3 with basis vectors $\{\mathbf{n}, \mathbf{m}, \tilde{\mathbf{m}}\}$ satisfying,

$$\mathbf{m} \cdot \mathbf{m} = 1, \quad \mathbf{m} \cdot \mathbf{n} = 0, \quad \tilde{\mathbf{m}} = \mathbf{n} \times \mathbf{m}.$$

By considering the action of $R(\mathbf{n}, \theta)$ on these basis vectors show that this matrix corresponds to a rotation through an angle θ about an axis parallel to \mathbf{n} and check that it is an element of $SO(3)$.

4. Show that the set of matrices

$$U = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 - |\beta|^2 = 1$ forms a group. How would you check that it is a Lie group? Assuming that it is a Lie group, determine its dimension. By splitting α and β into real and imaginary parts, consider the group manifold as a subset of \mathbb{R}^4 and show that it is non-compact. You may use the fact that a compact subset S of \mathbb{R}^n is necessarily bounded; in other words there exists $B > 0$ such that $|\mathbf{x}| < B$ for all $\mathbf{x} \in S$.

5. Show that any $SU(2)$ matrix U can be expressed in the form

$$U = \begin{pmatrix} \alpha & \beta \\ -\beta^* & \alpha^* \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Deduce that an alternative form for an $SU(2)$ matrix is

$$U = a_0 I + i \mathbf{a} \cdot \boldsymbol{\sigma}$$

with (a_0, \mathbf{a}) real, $\boldsymbol{\sigma}$ the Pauli matrices, and $a_0^2 + \mathbf{a} \cdot \mathbf{a} = 1$. Using the second form, calculate the product of two $SU(2)$ matrices.

6. Consider a real vector space V with product $* : V \times V \rightarrow V$. The product is bilinear and associative. In other words, for all elements $X, Y, Z \in V$ and scalars $\alpha, \beta \in \mathbb{R}$, we have

$$(\alpha X + \beta Y) * Z = \alpha X * Z + \beta Y * Z, \quad Z * (\alpha X + \beta Y) = \alpha Z * X + \beta Z * Y$$

and also $(X * Y) * Z = X * (Y * Z)$. Define the bracket of two vectors X and $Y \in V$ as the commutator,

$$[X, Y] = X * Y - Y * X$$

Show that, equipped with this bracket, V becomes a Lie algebra.

7. Verify that the set of matrices

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \quad a, b, c \in \mathbb{R}$$

forms a matrix Lie group, G . What is the underlying manifold of G ? Is the group abelian? Find the Lie algebra, $L(G)$, and calculate the bracket of two general elements of it. Is the Lie algebra simple?

8. A useful basis for the Lie algebra of $GL(n)$ consists of the n^2 matrices T^{ij} ($1 \leq i, j \leq n$), where $(T^{ij})_{\alpha\beta} = \delta_{i\alpha} \delta_{j\beta}$. Find the structure constants in this basis.

9. Let $\exp iH = U$. Show that if H is hermitian then U is unitary. Show also, that if H is traceless then $\det U = 1$. How do these results relate to the theorem that the exponential map $X \rightarrow \exp X$ sends $L(G)$, the Lie algebra of G , to G ?