Symmetries, Fields and Particles. Examples 3

1. In this question and the following ones, $\mathrm{L}(G)$ denotes the Lie algebra of a Lie group G and $\mathrm{L}_{\mathbb{C}}(G)$ denotes its *complexification*.

i) Show that

$$\mathrm{L}_{\mathbb{C}}\left(SU(2)\right) \simeq \mathrm{L}\left(SL(2,\mathbb{C})\right)$$

where the RHS is considered as a complex Lie algebra.

ii) By considering the subalgebra of real matrices show that $\mathrm{L}_{\mathbb{C}}\left(SU(2)\right)$ has two inequivalent real forms.

2. Write down a basis for a Cartan subalgebra of $\mathrm{L}_{\mathbb{C}}(SO(2n))$ and for a Cartan subalgebra of $\mathrm{L}_{\mathbb{C}}(SO(2n+1))$ for arbitrary integer n. Hence find the roots of the Lie algebras $\mathrm{L}_{\mathbb{C}}(SO(3))$ and $\mathrm{L}_{\mathbb{C}}(SO(4))$ and write down the corresponding step operators in your chosen basis.

3. If α is a root of a simple complex Lie algebra of finite dimension show that the only values of $k \in \mathbb{C}$ for which $k\alpha$ is a root are $k = \pm 1$

4. Starting from the constraint,

$$\frac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z} \qquad (*)$$

on the inner product of any two roots α and β, show that any complex simple Lie algebra of finite dimension has roots of at most two different lengths.

5.

i) Using the constraint (*) of question 3, show that the off-diagonal elements of the Cartan matrix A of a finite-dimensional complex simple Lie algebra of rank r obey

$$A_{ij}A_{ji} < 4 \quad \text{for all} \quad i \neq j = 1, 2, \ldots, r.$$

ii) Consider possible 3×3 Cartan matrices of the form,

$$\begin{pmatrix} 2 & l & m \\ l' & 2 & n \\ m' & n' & 2 \end{pmatrix}$$

Using the constraints on the Cartan matrix derived in the lectures together with the result of part i), show that l, m and n cannot all be non-zero, and find all the allowed Cartan matrices with $m = 0$. Show that your solutions exhaust all the rank 3 simple Lie algebras in the Cartan classification. Why do no additional solutions with $m \neq 0$ arise?

6. Find a set of simple roots for the matrix Lie algebra $\mathrm{L}_{\mathbb{C}}(SU(3))$ and determine the corresponding Cartan matrix showing that it coincides with the Cartan matrix of the Lie algebra A_2 in the Cartan classification.

7. Starting from its Dynkin diagram, construct the root system of the simple Lie algebra B_2.

8. The simple Lie algebra A_2 has simple roots α, β and a single additional positive root $\theta = \alpha + \beta$. Choose a basis for the the Lie algebra consisting of the generators $\{h^\alpha, h^\beta, e^{\pm\alpha}, e^{\pm\beta}, e^{\pm\theta}\}$. Write down all brackets between Cartan generators and step operators and also the brackets between step operators including normalisation constants for any brackets which are potentially non-zero. Determine the constraints on the normalisation constants which follow from the Jacobi identity. How could you fix any remaining ambiguities?

9. Determine the weights of the A_2 representation with Dynkin labels $(2,0)$.