Examples Sheet 4

1. A Lie algebra has simple roots $\alpha_1, \dots, \alpha_r$ and a diagonal Killing form. The fundamental weights satisfy
$$\frac{2(\omega_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{ij}.$$
   (a) Show that $\alpha_i = \sum_j A_{ij}\omega_j$ where $[A_{ij}]$ is the Cartan matrix.
   (b) A rank-2 Lie algebra has simple roots with coordinates $\alpha_1 = (1, 0)$ and $\alpha_2 = (-1, 1)$. What is the Cartan matrix?
   (c) Assuming any other positive roots are equal in length to either one of the simple roots, show that $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = 2\alpha_1 + \alpha_2$ are the other positive roots.
   (d) Draw the root diagram, and show that the dimension of the Lie algebra is 10.
   (e) Construct the fundamental weights ω_1, ω_2.
   (f) How is the highest weight of the representation whose weights coincide with the roots of the Lie algebra related to the fundamental weights?

2. Consider the Lie algebra with exactly 2 simple roots, $\alpha_1 = (1, 0)$ and $\alpha_2 = \frac{1}{2}(-3, \sqrt{3})$ (and a diagonal Killing form).
   (a) Determine the fundamental weights ω_1 and ω_2. Let $|q_1, q_2\rangle$ be a state corresponding to the weight $q_1\omega_1 + q_2\omega_2$.
   (b) Assuming $E_{\pm\alpha_i}, H_{\alpha_i}$ are the $SU(2)$ generators associated with the roots α_i construct a basis for the representation space starting from a highest weight vector (*i*) $|1, 0\rangle$ and (*ii*) $|0, 1\rangle$ by the successive action of $E_{-\alpha_1}$ and $E_{-\alpha_2}$ on the highest weight state.
   (c) Show that the dimensions of the space are respectively 7 and 14 (in the second case there are two independent states with $q_1 = q_2 = 0$).
   (d) Construct the weight diagram and in the 14-dimensional case show that it coincides with the root diagram.

3. A Lie algebra has a Cartan subalgebra $H = (H_1, \dots, H_r)$ and the remaining generators are E_α, corresponding to roots α, where $[H, E_\alpha] = \alpha E_\alpha$. Assume $[E_\alpha, E_{-\alpha}] = H_\alpha = 2(\alpha, H)/(\alpha, \alpha)$. For a root β, E_β satisfies
$$[E_\alpha, E_\beta] = 0\,, \quad [H_\alpha, E_\beta] = nE_\beta\,, \quad \underbrace{[E_{-\alpha}, [\dots, [E_{-\alpha}}_{r}, E_\beta]\dots]] = E_{\beta - r\alpha}\,.$$
   (a) Show that
$$[E_\alpha, E_{\beta - r\alpha}] = r(n - r + 1)E_{\beta-(r-1)\alpha}\,.$$

(b) Show that we may assume $E_{\beta-(n+1)\alpha} = 0$ for some integer n.

4. Decompose the following tensor product representations of $A_2 = \mathfrak{su}(3)$ into irreducible components: (a) $\mathbf{3} \otimes \bar{\mathbf{3}}$ and (b) $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3}$. Discuss the connections between these irreducible representations and the quark model of light mesons and baryons [c.f. Wikipedia]. You might also think about the spin-$\frac{3}{2}$ baryons Δ^{++} and Ω^{-}, whose quark content is, respectively, uuu and sss , and what Pauli's exclusion principle implies about the quantum numbers of those quarks.

5. Find the smallest-dimension, irreducible representation of B_2. Decompose the tensor product of two copies of this representation into irreps of B_2, giving the dimension of each component.

6. Consider a gauge theory whose gauge group G is a matrix Lie group. The corresponding gauge field,

$$A_\mu : \mathbb{R}^{1,3} \to L(G)\,,$$

transforms as

$$A_\mu \mapsto A'_\mu = g A_\mu g^{-1} - (\partial_\mu g) g^{-1}$$

under a gauge transformation

$$g : \mathbb{R}^{1,3} \to G\,.$$

For the case $G = SU(N)$, check that $A'_\mu(x)$ takes values in the Lie algebra $L(G)$. Explain why this is true for any matrix Lie group G. Writing $g = \exp \varepsilon X$, with $\varepsilon \ll 1$, show that the corresponding infinitesimal gauge transformation coincides with the one defined in the lectures.

7. For a group with a Lie algebra with a basis $\{T_a\}$ such that $[T_a, T_b] = f^c{}_{ab} T_c$ let $\kappa_{ab} = (T_a, T_b)$ where $(\ ,\ )$ is an invariant symmetric bilinear form so that $([X,Y],Z) = -(Y,[X,Z])$.

   (a) If D_μ is an appropriate covariant derivative involving a gauge field A^a_μ, verify

   $$\partial_\mu (X(x), Y(x)) = (D_\mu X(x), Y(x)) + (X(x), D_\mu Y(x)).$$

   (b) Let $T^\mu{}_\nu = (F^{\mu\sigma}, F_{\nu\sigma}) - \frac{1}{4}\delta^\mu{}_\nu (F^{\sigma\rho}, F_{\sigma\rho})$. Using the Bianchi identity, show that

   $$\partial_\mu T^\mu{}_\nu = (D_\mu F^{\mu\sigma}, F_{\nu\sigma})\,.$$

   (c) For a variation δA^a_μ obtain also $\delta \frac{1}{4}\epsilon^{\mu\nu\sigma\rho}(F_{\mu\nu}, F_{\sigma\rho}) = \partial_\mu \epsilon^{\mu\nu\sigma\rho}(\delta A_\nu, F_{\sigma\rho})$.

   (d) By letting $A_\mu \to t A_\mu$, differentiating with respect to t, then integrating, show that

   $$\frac{1}{4}\epsilon^{\mu\nu\sigma\rho}(F_{\mu\nu}, F_{\sigma\rho}) = \partial_\mu \epsilon^{\mu\nu\sigma\rho}\big(A_\nu, \partial_\sigma A_\rho + \frac{1}{3}[A_\sigma, A_\rho]\big)\,.$$

8. With notation as in the previous question define a 3-dimensional Lagrangian

$$\mathcal{L} = \epsilon^{\mu\nu\rho}\big(\kappa_{ab}A^a_\mu\partial_\nu A^b_\rho + \tfrac{1}{3}f_{abc}A^a_\mu A^b_\nu A^c_\rho\big).$$

For a gauge transformation $\delta A^a_\mu = -\partial_\mu\lambda^a - f^a{}_{bc}A^b_\mu\lambda^c$ show that $\delta\mathcal{L} = -\partial_\mu\big(\epsilon^{\mu\nu\rho}\kappa_{ab}\lambda^a\partial_\nu A^b_\rho\big)$ so that $\int \mathrm{d}^3x\,\mathcal{L}$ is invariant.

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