The Standard Model: Example Sheet 4

David Tong, March 2025

1a. The coupling constant g for an $SU(N_c)$ gauge theory, coupled to N_f massless Dirac fermions and N_s massless scalars, each in the fundamental representation, runs at one loop as

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} - \frac{1}{3(4\pi)^2}\left(11N_c - 2N_f - \frac{1}{2}N_s\right)\log\frac{\Lambda_{UV}^2}{\mu^2}$$

i) Show that the $SU(2)$ sector of the Standard Model is asymptotically free.

ii) Suppose that the Higgs boson did not exist. The value of the weak coupling constant $\alpha_W = g^2/4\pi$ is $\alpha_W(\mu) \approx 1/30$ at $\mu = 100$ GeV. At what scale $\Lambda_{\rm weak}$ would the theory become strongly coupled?

iii) The $SU(3)$ strong force becomes strongly coupled at $\Lambda_{\rm QCD}$, resulting in a quark condensate $\langle \bar{q}_{L\,i} q_{R\,j}\rangle \approx -\Lambda_{\rm QCD}^3\delta_{ij}$. If the Higgs boson didn't condense, what effect would this have on the weak force?

iv) At 100 GeV, the strong coupling constant takes value $\alpha_s \approx 0.1$. At what scale $\mu \gg 90$ GeV, does $\alpha_s = \alpha_W$?

1b. The beta function for a $U(1)$ gauge theory, coupled to massless Weyl fermions with charges Y_i and massless scalars with charges $\tilde{Y}_i$, is

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} + \frac{1}{3(4\pi)^2}\left(2\sum_i Y_i^2 + \sum_i \tilde{Y}_i^2\right)\log\frac{\Lambda_{UV}^2}{\mu^2}$$

Is hypercharge asymptotically free?

The value of the hypercharge coupling $\alpha_Y = g'^2/4\pi$ is $\alpha_Y(\mu) \approx 1/98$ at $\mu = 100$ GeV. At what scale μ does $\alpha_s(\mu) = \alpha_Y(\mu)$? At what scale does $\alpha_W(\mu) = \alpha_Y(\mu)$?

2. Suppose that a new particle is discovered that transforms in the **3** of $SU(2)$, the **6** of $SU(3)$, and with $U(1)$ hypercharge Y. What is its electric charge after electroweak symmetry breaking?

3. Show that covariant kinetic terms for fermions contain the electroweak couplings

$$\mathcal{L}_{\text{weak}} = -\frac{e}{\sqrt{2}\sin\theta_W}(W^+_\mu J^\mu_+ + W^-_\mu J^\mu_-) - \frac{e}{\sin\theta_W\ \cos\theta_W} Z^\mu J^Z_\mu - eA^\mu J^{\text{EM}}_\mu$$

where $e = g\sin\theta_W = g'\cos\theta_W$ and the Weinberg angle is $\tan\theta_W = g'/g$ and the various currents are

$$\begin{aligned}
J^{\text{EM}}_\mu &= \frac{2}{3}(\bar{u}_L\bar{\sigma}_\mu u_L + \bar{u}_R\sigma^\mu u_R) - \frac{1}{3}(\bar{d}_L\bar{\sigma}_\mu d_L + \bar{d}_R\sigma^\mu d_R) - (\bar{e}_L\bar{\sigma}_\mu e_L + \bar{e}_R\sigma^\mu e_R) \\
J^Z_\mu &= \frac{1}{2}(\bar{u}_L\bar{\sigma}_\mu u_L - \bar{d}_L\bar{\sigma}_\mu d_L + \bar{\nu}_L\bar{\sigma}_\mu\nu_L - \bar{e}_L\bar{\sigma}_\mu e_L) - \sin^2\theta_W\, J^{\text{EM}}_\mu \\
J^+_\mu &= \bar{u}_L\bar{\sigma}_\mu d_L + \bar{\nu}_L\bar{\sigma}_\mu e_L \quad \text{and} \quad J^-_\mu = \bar{d}_L\bar{\sigma}_\mu u_L + \bar{e}_L\bar{\sigma}_\mu\nu_L
\end{aligned}$$

4. If the Standard Model had N generations, how many physical parameters are there in the quark Yukawa couplings? How does this counting split between mass parameters, mixing angles, and complex phases?

Assume that neutrinos get their mass from a dimension 5 operator. If the Standard Model has N generations, how many physical parameters are there in the electron Yukawa couplings and the coefficients of this dimension 5 operator? How does this counting split between mass parameters, mixing angles, and complex phases?

5*. After electroweak symmetry breaking, the mass terms for two generations of quarks can be written in terms of *Dirac* fermions q^i and $\tilde{q}^i$ with $i = 1, 2$, as

$$\mathcal{L}_{\text{mass}} = -m_{ij}\bar{q}^i q^j - \tilde{m}_{ij}\bar{\tilde{q}}^i\tilde{q}^j$$

Suppose that the mass matrices take the form

$$m = \begin{pmatrix} 0 & a \\ a^\star & b \end{pmatrix} \quad \text{and} \quad \tilde{m} = \begin{pmatrix} 0 & c \\ c^\star & d \end{pmatrix}$$

with $a, c \in \mathbb{C}$ and $b, d \in \mathbb{R}$.

i) Show that by a suitable rephasing of the quark fields, we can take $a, c \in \mathbb{R}^+$.

ii) After this rephasing, let $R(\theta)$ and $R(\tilde{\theta})$ be the 2×2 rotation matrices that diagonalise the absolute values of the mass matrices, i.e.

$$R(\theta)\, m\, R(\theta)^{-1} = \text{diag}(m_1, -m_2) \quad \text{and} \quad R(\tilde{\theta})\, \tilde{m}\, R(\tilde{\theta})^{-1} = \text{diag}(\tilde{m}_1, -\tilde{m}_2)$$

Show that

$$\tan\theta = \sqrt{\frac{m_1}{m_2}} \quad \text{and} \quad \tan\tilde{\theta} = \sqrt{\frac{\tilde{m}_1}{\tilde{m}_2}}$$

iii) Show that the Cabibbo angle, which appears in the mixing matrix generated by the quarks' weak charged-current interactions, is given by $\theta_C = \theta - \tilde{\theta}$.

6*. Suppose that, in addition to the Standard Model Higgs H, transforming as $(\mathbf{2}, \mathbf{1})_{+1/2}$ under $U(1)_Y \times SU(2) \times SU(3)$, there is a second scalar field ϕ, transforming as $(\mathbf{2}, \mathbf{1})_{-1/2}$.

i) What are the electric charges of the components of ϕ?

ii) What is the most general, renormalisable potential involving ϕ and H consistent with gauge symmetry? How many independent real parameters does it contain?

iii) Suppose the parameters of the potential are such that it is minimised when

$$\langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{and} \quad \langle \phi \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} u + iw \\ 0 \end{pmatrix}$$

with $u, v, w \in \mathbb{R}$. Does the expectation value of ϕ break electromagnetism?

Compute the masses of the gauge bosons $W^{\pm}_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \mp iW^2_\mu)$ and suitable linear combinations of W^3_μ and B_μ. What are the charges of the remaining scalar fields after symmetry breaking?

iv) What are the possible Yukawa couplings of the fields, H and ϕ, to the fermions?

7. (**More challenging!**) Consider $SU(5)$ gauge theory coupled to two left-handed Weyl fermions transforming in the $\bar{\mathbf{5}}$ and $\mathbf{10}$ representations of the gauge group.

i) Show that the gauge anomaly cancels.

ii) Show that $G = U(1) \times SU(2) \times SU(3)$ is a subgroup of $SU(5)$. (Note: all generators should be normalised so that $\operatorname{Tr} T^A T^B = \frac{1}{2}\delta^{AB}$. You can use this to fix the overall normalisation of the $U(1)$ generator.)

Decompose the $\bar{\mathbf{5}}$ and $\mathbf{10}$ fermions into representations of G.

iii) Let ϕ be a scalar field in the adjoint representation of $SU(5)$ with scalar potential

$$V(\phi) = -m^2 \operatorname{Tr} \phi^2 + a \operatorname{Tr} \phi^4 + b \, (\operatorname{Tr} \phi^2)^2$$

with $m^2, a, b \in \mathbb{R}$. Assuming, without loss of generality, that $\langle \phi \rangle$ is diagonal, find the different symmetry breaking patterns. Show that there is a local minimum of $V(\phi)$ such that $SU(5)$ is broken to G provided that $7a + 30b > 0$.

iv) What is the (classical) prediction for the relative strengths of the gauge couplings for the $U(1)_Y$, $SU(2)$ and $SU(3)$ gauge groups at the scale of $\langle\phi\rangle$?

v) Draw a Feynman diagram in this model that shows that neither baryon number B nor lepton number L are conserved.