Part III Maths **Lent Term 2014**
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Advanced Quantum Field Theory
Example Sheet 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk $(*)$ may be more difficult.

1. $2n$ players apply to take part in a tennis tournament. If each player should play once in the first round, use a zero–dimensional QFT to compute how many different possible first round matches there are.

2. If B is an invertible $n\times n$ matrix and θ^i, $\bar\theta^i$, η_i and $\bar\eta_i$ are independent fermionic variables, show that

$$\int \mathrm{d}^n\theta\,\mathrm{d}^n\bar\theta\ \exp\left(\bar\theta^i B_{ij}\theta^j + \bar\eta_i\theta^i + \bar\theta^i\eta_i\right) = \det B\ \exp(\bar\eta_i (B^{-1})^{ij}\eta_j)$$

and that $\langle\theta^i\bar\theta^j\rangle = (B^{-1})^{ij}$.

3. Consider the partition function

$$\mathcal{Z}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{d}x\,\mathrm{e}^{-\frac{1}{2}x^2-\frac{\lambda}{4!}x^4} \qquad (\dagger)$$

for a zero–dimensional QFT with a quartic interaction with $\lambda > 0$.

(a) By expanding the integral in λ obtain the n^{th} order perturbative expansion

$$\mathcal{Z}_n(\lambda) = \sum_{\ell=0}^{n}\left(-\frac{\lambda}{4!}\right)^{\ell}\frac{(4\ell)!}{2^{2\ell}\,(2\ell)!\,\ell!}$$

and show for $\ell \leq 3$ that the coefficients a_ℓ of λ^ℓ in this expression are the sums of automorphism factors of the relevant loop Feynman graphs. (At two loops there is only one graph, at three loops there are two graphs and at four loops there are four.)

(b) *Optional but instructive:* Using any computer package, plot $\mathcal{Z}_n(\lambda = \frac{1}{10})$ against n to see that there is a region in n where $\mathcal{Z}_n$ appears to converge, before blowing up as n is increased.

(c) Show that the minimum value of $a_\ell \lambda^\ell$ occurs when $\ell \approx \frac{3}{2\lambda}$. Hence show that the Borel transform $\mathcal{B}\mathcal{Z}(\lambda) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\, a_\ell\, \lambda^\ell$ converges provided $|\lambda| < \frac{3}{2}$ and that in this case

$$\mathcal{Z}(\lambda) = \int_0^\infty \mathrm{d}z\, \mathrm{e}^{-z}\, \mathcal{B}Z(z\lambda)$$

so that $\mathcal{Z}(\lambda)$ may be recovered from its Borel transform.

(d) By expanding $\mathrm{e}^{-\frac{1}{2}x^2}$ in the integral (†) obtain the strong coupling expansion

$$\mathcal{Z}(\lambda) = \frac{1}{2\sqrt{\pi}} \sum_{L=0}^{\infty} \frac{(-1)^L}{L!}\, \Gamma\left(\frac{L}{2} + \frac{1}{4}\right) \left(\frac{6}{\lambda}\right)^{\frac{L}{2}+\frac{1}{4}}$$

for $\mathcal{Z}(\lambda)$ as a series in $1/\sqrt{\lambda}$. For $\lambda = \frac{1}{10}$ how many terms does one need to obtain the value at which the weak coupling expansion appeared to converge?

4. Consider the action $S(x,y) = \frac{m}{2}x^2 + \frac{M}{2}y^2 + \frac{\lambda}{4}x^2y^2$ describing two zero–dimensional fields x and y that are coupled by an interaction.

(a) Treating this as a zero–dimensional QFT, write down the Feynman rules for the propagators and the interaction.

(b) By considering connected diagrams, show that $\log(\mathcal{Z}/\mathcal{Z}_0) = -\frac{\lambda}{4mM} + \frac{\lambda^2}{4m^2M^2} + \mathcal{O}(\lambda^3)$, where $\mathcal{Z}_0 = \mathcal{Z}|_{\lambda=0}$.

(c) Integrate out the field y to show that the effective action for x is

$$\begin{aligned} S_{\text{eff}}(x) &:= -\log\left[\int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-S(x,y)}\right] \\ &= \frac{1}{2}\left(m + \frac{\lambda}{4M}\right)x^2 - \frac{\lambda^2}{16M^2}x^4 + \frac{\lambda^3}{48M^3}x^6 + \mathcal{O}(x^8) \end{aligned}$$

Which correlation functions can be computed using $S_{\text{eff}}(x)$ rather than $S(x,y)$?

(d) Working to order λ^2, compute $\frac{1}{2}\langle x^4\rangle$ first by using the original action $S(x,y)$ and then by using the effective action $S_{\text{eff}}(x)$. Check your results agree.

5. (∗) Let M be an $N \times N$ Hermitian matrix and consider the integral

$$\mathcal{Z}(a;N) = \int \mathrm{DM}\, \exp\left(-\frac{1}{2}\mathrm{tr}(\mathrm{M}^2) - \frac{a}{N}\mathrm{tr}(\mathrm{M}^4)\right)$$

where a is a coupling constant. Show that the action and measure

$$\mathrm{DM} := \prod_{i=1}^{N} \mathrm{dM}_{ii} \prod_{1\le i<j\le N} \mathrm{d}(\mathrm{Re}\,\mathrm{M}_{ij})\, \mathrm{d}(\mathrm{Im}\,\mathrm{M}_{ij})$$

are invariant under the transformation $\mathrm{M} \to \mathrm{M}' = U\mathrm{M}U^{-1}$ where U is a unitary matrix.

Show that $\mathcal{Z}(a;N)/\mathcal{Z}(0;N)$ has a perturbative expansion given by

$$\ln \frac{\mathcal{Z}(a;N)}{\mathcal{Z}(0;N)} = \sum_{g=0}^{\infty} N^{2-2g} \left(\sum_{n=0}^{\infty} (-a)^n F_{g,n} \right)$$

where $F_{g,n}$ is the number of ways to cover a genus g Riemann surface with n squares.

[For help with this question, you may wish to consult the first few sections of D. Bessis, C. Itzykson & B. Zuber, *Quantum Field Theory Techniques in Graphical Enumeration*, Adv. Applied Maths **1**, 109-157, (1980).]

6. For a quantum mechanical harmonic oscillator of mass m and frequency ω,

$$\langle x|\Phi_t|y\rangle = \sqrt{\frac{m\omega}{2\pi \mathrm{i}\, \sin \omega t}}\, \exp\left(\mathrm{i} m\omega \frac{(x^2+y^2)\cos\omega t - 2xy}{2\sin\omega t} \right)$$

is the amplitude to move from x to y during time t. Verify that for $\mathrm{Re}(\beta) > 0$

$$\mathrm{tr}(\mathrm{e}^{-\beta H}) = \int_{\mathbb{R}} \mathrm{d}x\, \langle x|\Phi_{-\mathrm{i}\beta}|x\rangle \qquad \text{and} \qquad \mathrm{tr}(P\mathrm{e}^{-\beta H}) = \int_{\mathbb{R}} \mathrm{d}x\, \langle -x|\Phi_{-\mathrm{i}\beta}|x\rangle\,,$$

where H is the usual harmonic oscillator Hamiltonian and P is the parity operator $P : x \to -x$. (The trace operator tr is defined by $\mathrm{tr}\,\mathcal{O} = \sum_n \langle n|\mathcal{O}|n\rangle$ where $\{|n\rangle\}$ is a complete set of orthonormal states.)

7. In a 1d QFT, let x be a real bosonic scalar field and let ψ and $\bar\psi$ be fermionic fields. Consider the action

$$S = \int \mathrm{d}\tau \left[\frac{1}{2}\left(\frac{\partial x}{\partial \tau}\right)^2 + \bar\psi \frac{\partial \psi}{\partial \tau} + \frac{1}{2}\lambda^2 h'(x)^2 - \lambda\, h''(x)\, \bar\psi\psi \right]$$

where λ is a coupling constant and $h(x)$ is a smooth function of $x(\tau)$ (and h' the derivative of this function).

(a) Show that this action is invariant under the transformations

$$\delta x = \epsilon\bar\psi - \bar\epsilon\psi$$
$$\delta\psi = \epsilon(-\dot x + \lambda\, h'(x))$$
$$\delta\bar\psi = \bar\epsilon(\dot x + \lambda\, h'(x))$$

where ϵ and $\bar\epsilon$ are constant fermionic parameters and $\dot x = \partial x/\partial\tau$. Find the conserved charges Q and $\bar Q$ associated to these two symmetries.

(b) Show that $Q\bar Q + \bar Q Q = 2H$ where H is the Hamiltonian associated to the above Lagrangian. Hence or otherwise show that the energies of the system are non–negative, and that the ground state $|\Psi\rangle$ obeys $Q|\Psi\rangle = \bar Q|\Psi\rangle = 0$.

(c) ($*$) Put this theory on a circle and take $x(\tau)$, $\psi(\tau)$ and $\bar{\psi}(\tau)$ each to be periodic. Show that the partition function $\mathcal{Z} = \int \mathrm{D}x\, \mathrm{D}\psi\, \mathrm{D}\bar{\psi}\; \mathrm{e}^{-S}$ with these boundary conditions is independent both of the radius of the circle and of λ, provided $\lambda \neq 0$. [*Hint: Show that the operator you obtain by differentiating* e^{-S} *w.r.t.* λ *can be written as* $\{Q, \chi\}$ *for some* χ.]

(d) What is the operator expression to which this partition function corresponds?

(e) ($*$) Argue that the only field configurations that contribute to $\mathcal{Z}$ have $\dot{x} = h'(x) = 0$. By expanding the action in a neighbourhood of these configurations, evaluate $\mathcal{Z}$ in the case that $h(x)$ is a polynomial of degree n with isolated roots.