Part III Maths **Lent Term 2014**
David Skinner d.b.skinner@damtp.cam.ac.uk

Advanced Quantum Field Theory
Example Sheet 2

Please email me with any comments about these problems, particularly if you spot an error.

1. Consider the partition function

$$Z = \int_{\mathbb{R}^n} \frac{\mathrm{d}^n x}{(2\pi)^{n/2}} \exp\left(-\frac{1}{2}x_i A_{ij} x_j - V(x_i)\right)$$

of a zero–dimensional QFT for n real variables x_i with a polynomial interaction $V(x_i)$. You may assume that V is at least cubic in the x_is, but has an arbitrarily high degree.

(a) Use topological identities to show that an ℓ-loop connected vacuum diagram that involves r propagators and s_n n-particle vertices must have $r - \sum_n s_n = \ell - 1$ and $2r = \sum_n ns$.

(b) Hence draw all three–loop connected vacuum diagrams that are one particle irreducible. (There are 8 diagrams in all; two of these involve only 3-particle vertices, two involve only 4-particle vertices and one involves just a 6-particle vertex.) What are the symmetry factors for each diagram?

(c) Couple the above action to sources J_i and compute the partition function in terms of a differential operator acting on $\exp(-V(J_i))$. Use this formula to check your previous results by expanding your expression for the partition function.

2. Consider the theory given by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4 \ .$$

where ϕ is a real scalar field.

(a) Determine all connected one loop graphs, complete with their appropriate symmetry factors, which contribute to $\langle\phi(x)\phi(y)\rangle$, to $\langle\phi(x)\phi(y)\phi(z)\rangle$ and to $\langle\phi(x)\phi(y)\phi(z)\phi(w)\rangle$, expressing your answer in terms of integrals over D-dimensional loop momenta. [*You are not required to evaluate the integrals.*]

(b) Now set $\lambda = 0$ so that just the cubic interaction remains. Determine the momentum space correlation function $\int \prod_{i=1}^{3} \mathrm{d}^D x_i \, \mathrm{e}^{\mathrm{i}p_i \cdot x_i} \langle \phi(x_1)\phi(x_2)\phi(x_3) \rangle$ to one loop accuracy.

3. Show that the charge conjugation transformation

$$\psi \to \psi' := C\bar{\psi}^{\mathrm{T}} \qquad \text{and} \qquad A_\mu \to A'_\mu = -A_\mu \,,$$

where C is any matrix obeying $C^{-1}\gamma^\mu C = -\gamma^{\mu\mathrm{T}}$, leaves the QED action invariant. By considering the QED partition function in the presence of an external source term $\int \mathrm{d}^D x \, J^\mu(x) A_\mu(x)$, show that the correlation function of an odd number of photon fields (and no other operators) vanishes exactly. [*Assume that the path integral measure is invariant under charge conjugation.*]

4. In flat space, a *special conformal transformation* is given by

$$x^\mu \to x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b \cdot x + b^2 x^2}$$

where b^μ is a constant parameter and where the distances are computed using the flat space metric $\eta_{\mu\nu}$.

(a) Show that this transformation is generated by the operator $K^\mu(x) = \mathrm{i}(x^2\eta^{\mu\nu} - 2x^\mu x^\nu)\partial_\nu$

(b) Show that the classical action

$$S = \int \mathrm{d}^4 x \left[\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathrm{i}\bar{\psi} \not{D} \psi \right]$$

describing QED with a massless fermion of charge q is invariant under special conformal transformations.

(c) Obtain the corresponding Ward identity in terms of the stress energy tensor.

5. The Lagrangian

$$\mathcal{L}_{\mathrm{QED}} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(\mathrm{i}\not{D} + m)\psi$$

for quantum electrodynamics is invariant under the transformation $\psi \to \mathrm{e}^{\mathrm{i}\alpha}\psi$, $\bar{\psi} \to \bar{\psi}\,\mathrm{e}^{-\mathrm{i}\alpha}$ and $A_\mu \to A_\mu$ where α is a constant and where $D_\mu = \partial_\mu - \mathrm{i}qA_\mu$.

(a) By allowing α to depend on position, find the corresponding conserved current $J_\mu(x)$.

(b) Obtain the Ward identity relating the derivative of $\langle J_\mu(x) \prod_{i=1}^{n} \mathcal{O}_i(y_i) \rangle$ to variations of the operators $\mathcal{O}_i$.

(c) In the case that the only operator insertions are single Dirac fermions $\psi(y_1)$ and $\bar{\psi}(y_2)$, show that in momentum space, the Ward identity implies $(p-k)_\mu \Gamma^\mu(k,p) = \mathrm{i}\left(S(k)^{-1} - S(p)^{-1}\right)$, where $\mathrm{i}S(p)$ is the exact Dirac propagator in momentum space, and where the vertex function $\Gamma^\mu(k,p)$ is defined by

$$\delta^D(\ell{+}p{-}k)\, S(k)\Gamma^\mu(k,p)S(p) := \frac{\mathrm{i}}{q}\int \mathrm{d}^D x\, \mathrm{d}^D y_1\, \mathrm{d}^D y_2\, \mathrm{e}^{-\mathrm{i}\ell\cdot x}\, \mathrm{e}^{-\mathrm{i}k\cdot y_1}\, \mathrm{e}^{+\mathrm{i}p\cdot y_2}\, \langle J^\mu(x)\, \psi(y_1)\bar{\psi}(y_2)\rangle\ .$$

where $J^\mu(x)$ is the current found above.

(d) Taking the limit of this equation as $p_\mu \to k_\mu$ to express $\Gamma^\mu(k,k)$ in terms of the electron self–energy $\Sigma(\not{k})$. Use the renormalization conditions of $\Sigma(\not{k})$ to show that $\bar{u'}_k\Gamma^\mu(k,k)u_k = \bar{u'}_k\gamma^\mu u_k$ where u_k and u'_k are on–shell Dirac wavefunctions.

6. Obtain the Källen–Lehmann representation of the exact 2-pt function of a complex scalar field. [*Hint: To show that the spectral function is real, you should require that the expecation value of the commutator* $[\phi(x),\phi(y)]$ *vanishes when* x *and* y *are space–like separated.*]

7. Derive the Källen–Lehmann representation of the exact two point function $\langle\psi(x)\bar{\psi}(y)\rangle$ of a fermionic Dirac spinor.

8. Write down all possible Lorentz invariant renormalizable (or super–renormalizable) Lagrangians for a theory of a complex scalar field coupled to a Dirac fermion in $D = 2$, $D = 3$, $D = 4$, $D = 6$ and $D = 10$.

9. Compute the 1-loop approximation to the vertex shift $\delta\lambda$ in massive $\lambda\phi^4$ theory by working strictly in $D = 4$ but imposing a sharp cutoff $p^2 \leq \Lambda^2 \gg m^2$ on the Euclidean loop integrals. Defining the physical coupling λ to be the value of the 4-point amplitude at the symmetric point $s = t = u = -4\mu^2/3$, show explicitly that the momentum cutoff Λ cancels out of this scattering amplitude at arbitrary external momenta. Show further that this amplitude agrees with the amplitude computed during the lectures using dimensional regularization.

10. Consider the theory of a (bare) scalar field $\phi_0(x)$ and (bare) fermionic Dirac field ψ_0, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\,\partial^\mu\phi_0 + \frac{1}{2}m_0^2\,\phi_0^2 + \bar{\psi}_0(\mathrm{i}\not{\partial} + \mu_0)\psi_0 + g_0\,\phi_0\bar{\psi}_0\psi_0$$

in Euclidean signature. Write the Lagrangian in terms of renormalized fields and counterterms for the fields, masses and coupling g_0. Use dimensional regularization to evaluate these counterterms to 1-loop accuracy, stating the renormalization conditions you use. Show that all 1-loop amplitudes are finite when expressed in terms of the renormalized couplings.