

Examples Sheet 2

1. Consider the theory given by the action

$$S[\phi] = \int d^d x \left[\frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} m^2 \phi^2 + \frac{g}{3!} \phi^3 + \frac{\lambda}{4!} \phi^4 \right]$$

where ϕ is a real scalar field. Determine all one-loop 1PI graphs, complete with their appropriate symmetry factors, which contribute to

$$\langle \phi(x)\phi(y) \rangle, \quad \langle \phi(x)\phi(y)\phi(z) \rangle, \quad \text{and} \quad \langle \phi(x)\phi(y)\phi(z)\phi(w) \rangle,$$

expressing your answers in terms of integrals over d -dimensional loop momenta. [You are not required to evaluate the integrals.]

2. Consider a scalar field ϕ with potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{6}\mu^{\epsilon/2}g(\mu)\phi^3$ in dimension $d = 6 - \epsilon$. Here μ is an arbitrary mass scale introduced so that the coupling $g(\mu)$ is dimensionless.

- (a) Draw the one-loop 1PI graph which contributes to the propagator at order g^2 .
 (b) Using dimensional regularization, show that the divergent part of the corresponding integral for the 6-dimensional theory is

$$-\frac{1}{\epsilon} \frac{g^2}{(4\pi)^3} \left(m^2 + \frac{p^2}{6} \right)$$

where p is the external momentum. Also compute the divergence corresponding to the 1PI one-loop graph that gives a g^3 correction to the 3-point function, and find the one-loop divergence for the 1-point function.

[Hint: use the Feynman parametrization $\frac{1}{AB} = \int_0^1 dx \int_0^1 dy \frac{\delta(x+y-1)}{(xA+yB)^2}$, then change variables to get denominators of integrands into the form $(\ell^2 + \Delta)^2$, where ℓ is the loop momentum. You can generalize, introducing more Feynman parameters for integrals which need it.]

- (c) Show that in 6 dimensions all these divergences may be cancelled by introducing the counterterm action

$$S_{\text{ct}}[\phi] = \int d^d x \mathcal{L}_{\text{ct}} = -\frac{1}{\epsilon} \frac{1}{6(4\pi)^3} \int d^d x \left[\frac{g^2}{2} \partial_\mu \phi \partial^\mu \phi + \mu^{-\epsilon} (V''(\phi))^3 \right].$$

Check that \mathcal{L}_{ct} has dimension d .

- (d) Determine the β -function for the coupling g and show $\beta(g) < 0$ at small g .

3. Show that under the redefinition $g_i \mapsto g'_i(g_j)$ of the couplings of a theory at scale Λ , the β -functions transform as

$$\beta_i \mapsto \beta'_i = \frac{\partial g'_i}{\partial g_j} \beta_j.$$

Show that in a theory with a single coupling g , the first two terms in the β -function $\beta(g) = ag^3 + cg^5 + \mathcal{O}(g^7)$ are invariant under any coupling constant redefinition of the form $g \mapsto g' = g + \mathcal{O}(g^3)$.

Show that, in a neighbourhood of $g = 0$, it is possible to choose this redefinition so as to remove all terms *except* these first two. [*Hint: consider setting $g'(g) = g + g^3 f(g)$ where $f(0) = 1$.*] What is the significance of this calculation?

4. Consider a four-dimensional theory whose only couplings are a mass parameter m^2 and a marginally relevant coupling g .
- Write down generic expressions for the β -functions in such a theory to lowest non-trivial order. (You should be able to identify the *values* of the classical contributions to the β -functions and the *sign* of the leading-order quantum correction to $\beta(g)$.)
 - Sketch the RG flows for this theory.
 - Suppose that $g(\Lambda') = 0.1$ when the cut-off Λ' is fixed at 10^5 GeV. If $m^2(\Lambda')$ is measured to be $(100 \text{ GeV})^2$, what value of $m^2(\Lambda)$ would be needed at the higher scale $\Lambda = 10^{19}$ GeV?
 - Suppose you changed your value of $m^2(\Lambda)$ by one part in 10^{20} . What would be the change in $m^2(\Lambda')$?

Please e-mail me at M.Wingate@damtp.cam.ac.uk with any comments, especially any errors.