Part III Maths **Lent Term 2014**
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Advanced Quantum Field Theory
Example Sheet 3

Please email me with any comments about these problems, particularly if you spot an error.

1. With $c, c' > 0$ let
$$I_1(c) = \int_0^\infty \mathrm{d}x\, \frac{x^{1-\epsilon}}{x+c}\,, \qquad I_2(c,c') = \int_0^\infty \mathrm{d}x\, \frac{x^{1-\epsilon}}{(x+c)(x+c')}\,,$$
$$I_3(c) = \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \frac{x^{1-\epsilon}\, y^{1-\epsilon}}{(x+c)(y+c)(x+y+c)}\,.$$
For what values of ϵ are these integrals convergent?
Using $x^{-\lambda}\Gamma(\lambda) = \int_0^\infty \mathrm{d}\alpha\, \alpha^{\lambda-1} e^{-\alpha x}$ for $x > 0$ and also the identity $\Gamma(1-\epsilon)\Gamma(\epsilon) = \pi/\sin\pi\epsilon$ evaluate the integrals to obtain
$$I_1(c) = -\frac{\pi}{\sin\pi\epsilon}\, c^{1-\epsilon}\,, \qquad I_2(c,c') = \frac{\pi}{\sin\pi\epsilon}\, \frac{c^{1-\epsilon} - c'^{1-\epsilon}}{c-c'}\,,$$
$$I_3(c) = \Gamma(1-\epsilon)^2 \left(\Gamma(2\epsilon-1) - \Gamma(\epsilon)^2\right) c^{1-2\epsilon}\,.$$
Determine the divergent parts of I_1, I_2 and I_3 as given by poles in ϵ. Note that for $\epsilon = 0$, I_3 has subdivergences when $x \to \infty$ or $y \to \infty$. Explain why the subdivergences may be subtracted by considering
$$I_3(c) - \frac{2}{\epsilon}\, I_1(c) \sim \frac{1}{\epsilon^2}\left(1 - \frac{\epsilon}{2}\right) c\,,$$
which does not have any $\ln c$ divergent terms.

2. Using $(x^2)^{-\lambda}\,\Gamma(\lambda) = \int_0^\infty \mathrm{d}\alpha\, \alpha^{\lambda-1}\, \mathrm{e}^{-\alpha x^2}$, evaluate $\int \mathrm{d}^D x\, (x^2)^{-\lambda}\, \mathrm{e}^{\mathrm{i}p\cdot x}$. Check that the result is consistent with the standard Fourier inversion formula. Let $G_0(x)$ be the Green function whose Fourier transform is $1/p^2$. Calculate $\int \mathrm{d}^D x\, G_0(x)^n\, e^{\mathrm{i}p\cdot x}$ for $n = 2, 3$. Show that the poles in $\epsilon = 4 - D$ are in exact agreement with one and two loop momentum space calculations. What happens for $D \approx 3$? Why should we consider $n = 3, 4, 5$ in this case?

3. For a Lagrangian $\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\,\phi^2 - \frac{1}{6}g\,\phi^3$ determine the lowest order contribution to the correlation function $\prod_{i=1}^3 \int \mathrm{d}^D x_i\, \mathrm{e}^{\mathrm{i}p_i\cdot x_i}\, \langle\phi(x_1)\phi(x_2)\phi(x_3)\rangle$. If $\hat\phi(x)$ is the operator quantum field and $|p\rangle$ is the associated single particle state with $-p^2 = m^2$ show that to this order
$$\langle 0|\hat\phi(0)|p_1, p_2\rangle = -\frac{g}{(p_1+p_2)^2 + m^2}\,.$$

4. Let $\tau_n(p_1,\dots,p_n)$ be the one particle irreducible functions in four dimensions with n external lines with incoming momenta p_i after removal of a factor $i(2\pi)^4\delta(\sum p_i)$. In ϕ^4 theory, as calculated using dimensional regularisation in the $\overline{MS}$ scheme, show that to one loop we may obtain the finite results

$$\tau_2(p,-p) = -p^2 - m^2 + \lambda\frac{m^2}{32\pi^2}\left(1 - \ln\frac{m^2}{\mu^2}\right) ,$$

$$\tau_4(p_1,p_2,p_3,p_4) = -\lambda - \lambda^2\frac{1}{32\pi^2}\left(f(s)+f(t)+f(u)\right) ,$$

where

$$f(s) = \int_0^1 \mathrm{d}x \, \ln\frac{m^2 - x(1-x)s}{\mu^2} ,$$

and where $s=-(p_1{+}p_2)^2$, $t=-(p_1{+}p_4)^2$ and $u=-(p_1{+}p_3)^2$ are the usual Mandelstam variables.

Show that $\tau_2(p,-p)$ and $\tau_4(p_1,p_2,p_3,p_4)$ satisfy the RG equations

$$\left(\mu\frac{\partial}{\partial\mu} + \frac{3\lambda^2}{16\pi^2}\frac{\partial}{\partial\lambda} + \frac{\lambda m^2}{16\pi^2}\frac{\partial}{\partial m^2}\right)\tau_n = \begin{cases} \mathrm{O}(\lambda^2), n=2; \\ \mathrm{O}(\lambda^3), n=4 \end{cases}$$

For $s=t=u$ and $-s \gg m^2$ in τ_4 verify that

$$\tau_4(p_1,p_2,p_3,p_4) = -\left(\frac{1}{\lambda} - \frac{3}{32\pi^2}\ln\left(\frac{-s}{\mu^2}\right)\right)^{-1}$$

satisfies the equation with zero right hand side.

Suppose $\tau_4(p,p,-p,-p) = -\lambda'$ for $p^2=-m^2$ is an alternative definition of the coupling. Find λ' in terms of λ and express τ_4 in terms of λ' to $\mathrm{O}(\lambda'^2)$. Note that τ_4 is then independent of μ but that the limit $m^2 \to 0$ is singular.

5. Consider a scalar field ϕ with potential $V(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{6}\mu^{\epsilon/2}g\phi^3$ in dimension D and $\epsilon = 6-D$. Here μ is an arbitrary mass scale so that g is dimensionless. Draw the one-loop one particle irreducible graph which contributes to the propagator at order g^2.

Using dimensional regularisation, show that the divergent part of the corresponding integral for the six dimensional theory is

$$-\frac{1}{\epsilon}\frac{g^2}{(4\pi)^3}\left(m^2 + \frac{1}{6}p^2\right) ,$$

where p is the external momentum. Also compute the divergence corresponding to the one particle irreducible one-loop graph that gives a g^3 correction to three point function, and find the one loop divergence for the one point function.

Show that in six dimensions these divergences may be cancelled by introducing the one loop counterterm Lagrangian

$$\mathcal{L}_{\rm ct}(\phi) = \frac{1}{\epsilon}\frac{1}{6(4\pi)^3}\left(\frac{1}{2}g^2(\partial\phi)^2 + \mu^{-\epsilon}V''(\phi)^3\right).$$

Check that $\mathcal{L}_{\rm ct}$ has dimension D.

6. For the six dimensional ϕ^3 theory of the previous question express the bare g_0, m_0^2 in terms of the dimensionless coupling g and m^2 and also an arbitrary scale mass μ to lowest order. Determine the beta function $\beta(g)$ and also $\gamma_{m^2}(g)$ to lowest order and show that $\beta(g) < 0$ for g small.

7. For a theory with multiple couplings g_i the beta function $\beta_i(g) = \mu\frac{\partial}{\partial\mu}g_i$ defines a vector field. Show that under a redefinition $g_i \to g'_i(g_j)$ we have

$$\beta'_i(g') = \frac{\partial g'_i}{\partial g_j}\beta_j(g).$$

Show that for a single coupling with $\beta(g) = b_1 g^3 + b_2 g^5 + \mathrm{O}(g^7)$ and $g'(g) = g + \mathrm{O}(g^3)$ the first two terms in the beta function are invariant. Show also that it is possible to choose $g'(g)$ so that all terms other than the first two are zero.