Part III Maths **Lent Term 2014**
David Skinner d.b.skinner@damtp.cam.ac.uk

Advanced Quantum Field Theory
Example Sheet 4

Please email me with any comments about these problems, particularly if you spot an error.

1. Let t_A be the generators of a Lie algebra $\mathfrak{g}$, $[t_A, t_B] = \mathrm{i} f_{AB}^C t_C$, and let c^A be anticommuting variables. Show that

$$Q := c^A t_A - \frac{1}{2} f_{BC}^A c^B c^C \frac{\partial}{\partial c^A}$$

satisfies $Q^2 = 0$. Suppose $t_A = 0$ and also that $f_{ABC} = k_{CD} f_{AB}^D$ is completely antisymmetric, where k_{CD} is the Killing form on $\mathfrak{g}$. If $X = f_{ABC}\, c^A c^B c^C$ show that $QX = 0$ but that $X \neq QY$.

2. Consider a gauge-fixed action for a free (Abelian) gauge field A_μ of the form

$$S = \int \mathrm{d}^D x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + h\, \partial^\mu A_\mu + \frac{\xi}{2} h^2 + \bar{c}\, \partial^2 c \right)$$

where h is an auxiliary bosonic field and $(c, \bar{c})$ are anticommuting ghost and antighost fields.

(a) Verify that this action is invariant under the BRST transformations $\delta A_\mu = \epsilon \partial_\mu c$, $\delta c = 0$, $\delta \bar{c} = -\epsilon h$, $\delta h = 0$ and that δ is nilpotent.

(b) Show that the action can be written in the form

$$S = -\int \mathrm{d}^D x \left(\frac{1}{2} \Phi^{\mathrm{T}} \Delta \Phi + \bar{c}(-\partial^2) c \right)$$

where $\Phi = \begin{pmatrix} A_\mu \\ h \end{pmatrix}$, Φ^{T} is its transpose and where

$$\Delta = \begin{pmatrix} -\partial^2 \delta^\mu_{\ \nu} + \partial^\mu \partial_\nu & \partial_\nu \\ -\partial^\mu & -\xi \end{pmatrix} .$$

(c) Obtain equations for the normalized correlation functions $\langle \Phi(x)\,\Phi(0)^{\mathrm{T}}\rangle$ and $\langle c(x)\,\bar{c}(0)\rangle$ and show that

$$\int \mathrm{d}^D x\, \mathrm{e}^{-\mathrm{i}p\cdot x}\langle \Phi(x)\,\Phi(0)^{\mathrm{T}}\rangle = -\frac{\mathrm{i}}{p^2}\begin{pmatrix} \delta_\mu{}^\nu - (1-\xi)\frac{p_\mu p^\nu}{p^2} & \mathrm{i}p_\mu \\ -\mathrm{i}p^\nu & 0 \end{pmatrix}$$
$$\int \mathrm{d}^D x\, \mathrm{e}^{-\mathrm{i}p\cdot x}\langle c(x)\,\bar{c}(0)\rangle = -\frac{\mathrm{i}}{p^2}\,.$$

(d) Assuming that $\langle \delta Y\rangle = 0$ for any Y, consider $\langle \delta\,(\Phi(x)\,\bar{c}(0))\rangle$ and show that we must have $\langle h(x)\,h(0)\rangle = 0$. Obtain also a relation between $\langle c(x)\,\bar{c}(0)\rangle$ and $\langle A_\mu(x)\,h(0)\rangle$ which should be verified.

3. For a gauge theory coupled to scalars the single particle states are

$$|A_\mu^A(p)\rangle\,, \qquad |\phi_i(p)\rangle\,, \qquad |c^A(p)\rangle\,, \qquad |\bar{c}^A(p)\rangle\,,$$

where A runs over a basis of the adjoint representation and i similarly indexes the R representation. These states have non-zero scalar products

$$\langle A_\mu^A(p)|A_\nu^B(p')\rangle = \eta_{\mu\nu}\delta^{AB}\delta_{pp'} \qquad \langle \phi_i(p)|\phi_j(p')\rangle = \delta_{ij}\delta_{pp'}$$
$$\langle c^A(p)|\bar{c}^B(p')\rangle = \langle \bar{c}^A(p)|c^B(p')\rangle = \delta^{AB}\delta_{pp'}$$

where $\delta_{pp'} := (2\pi)^{D-1}\,2p^0\,\delta^{(D-1)}(\mathbf{p}-\mathbf{p}')$. The non-zero action of the BRST charge Q is given by

$$Q|A_\mu^A(p)\rangle = \alpha p_\mu |c^A(p)\rangle\,, \quad Q|\phi_i(p)\rangle = \sum_A v_{iA}|c^A(p)\rangle$$
$$Q|\bar{c}^A(p)\rangle = \beta p^\mu |A_\mu^A(p)\rangle + \sum_i \bar{v}_{Ai}|\phi_i(p)\rangle$$

while the ghost charge Q_{gh} acts non-trivially as

$$Q_{\mathrm{gh}}|c^A(p)\rangle = \mathrm{i}|c^A(p)\rangle\,, \qquad Q_{\mathrm{gh}}|\bar{c}^A(p)\rangle = -\mathrm{i}|\bar{c}^A(p)\rangle\,.$$

Verify that this is compatible with Q and Q_{gh} being Hermitian if α, β, v_{iA} and $\bar{v}_{Ai}$ are related appropriately. Assume a basis has been chosen so that $\sum_i \bar{v}_{Ai}v_{iB} = \delta_{AB}\rho_A$ and so is diagonal. Find the conditions under which the BRST charge $Q^2 = 0$. Use this to determine the possible physical single particle states.

4. Consider a gauge invariant Lagrangian density of the form

$$\mathcal{L}(A) = -\frac{1}{4}\mathrm{tr}\left(F^{\mu\nu}X(D^2)F_{\mu\nu}\right)\,,$$

where $D_\lambda F_{\mu\nu} = \partial_\lambda F_{\mu\nu} + [A_\lambda, F_{\mu\nu}]$ and where $X(D^2) = 1 + (-D^2)^r/\Lambda^{2r}$ for some scale Λ. The full quantum Lagrangian with gauge fixing and ghost fields is

$$\mathcal{L}_q(A,c,\bar{c}) = \mathcal{L}(A) - \frac{1}{2\xi}\mathrm{tr}\left(\partial^\mu A_\mu\, X(\partial^2)\,\partial^\nu A_\nu\right) + \mathrm{tr}\left(\bar{c}\,X(\partial^2)\,\partial^\mu D_\mu c\right)\,.$$

(a) Show that the Feynman rules require that the gauge and ghost propagators are each proportional to p^{-2-2r}.

(b) Show that there must be vertices with n gauge field legs $n = 3, \dots 2r + 4$ and with $2r + 4 - n$ powers of momentum, but that there is just a single vertex involving both ghost and gauge fields with $2r + 1$ momentum factors.

(c) Hence show that, in four dimensions, the superficial degree of divergence of an ℓ-loop Feynman graph with E_A external gauge field lines and E_{gh} ghost field lines is

$$\delta = 4 - E_A - E_{\text{gh}} - 2r(\ell - 1)\,.$$

5. Consider pure (= no charged matter) electrodynamics with Lagrangian $\mathcal{L} = -\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu}$. Let $\mathrm{W}_\gamma[A]$ be a Wilson loop around a closed curve γ.

(a) Show that

$$\langle \mathrm{W}_\gamma[A] \rangle = \exp\left[-\frac{e^2}{8\pi^2} \oint_\gamma \mathrm{d}x^\mu \oint_\gamma \mathrm{d}y_\mu \, \frac{1}{(x-y)^2} \right] .$$

(b) Now suppose γ is a large rectangle with space-like width L and time-like length T. Compute $\langle \mathrm{W}_\gamma[A] \rangle$ in the limit $T \gg L$. By comparing your result to the usual expression for time evolution, show that the potential between two point-like charges at fixed separation L in electrodynamics is $V(L) = -e^2/4\pi L$.

(c) In Feynman gauge, the propagator for a non-Abelian gauge field is

$$\langle A^B_\mu(x) A^C_\nu(y) \rangle = \mathrm{i}\eta_{\mu\nu}\, \delta^{BC} \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, \frac{\mathrm{e}^{-\mathrm{i}p\cdot(x-y)}}{p^2} .$$

Compute the expectation value of a Wilson loop in pure $\mathrm{SU}(N)$ Yang-Mills theory to lowest non-trivial order in the coupling g^2. [*Your result should depend on a choice of the representation R of $\mathfrak{su}(N)$.*]

(d) Show that, to this order, the Coulomb potential of non-Abelian gauge theory is $V(L) = -g^2 C_2(R)/4\pi L^2$, where $C_2(R)$ is the quadratic Casimir of the R representation.

6. For a beta function $\beta(g) = -b_1 g^3 - b_2 g^5 + \mathrm{O}(g^7)$ show that the solution for the running coupling can be expressed in the form

$$\frac{1}{g^2} = b_1 \ln \frac{\mu^2}{\Lambda^2} + \frac{b_2}{b_1} \ln\left(\ln \frac{\mu^2}{\Lambda^2} \right) + \mathcal{O}\left(1 \Big/ \ln \frac{\mu^2}{\Lambda^2} \right)$$

or equivalently

$$\Lambda^2 = \mu^2 \, \mathrm{e}^{-\frac{1}{b_1 g^2}} \left(b_1 g^2 \right)^{-\frac{b_2}{b_1^2}} \left(1 + \mathcal{O}(g^2) \right) .$$

Suppose $\bar{g} = g + v g^3 + \mathcal{O}(g^5)$. Show that $\bar{g}^2$ can be expresssed in terms of $\bar{\Lambda}^2$ as above where $\ln \bar{\Lambda}^2/\Lambda^2 = v/b_1$.