String Theory: Example Sheet 3

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1. Take the action for the ghost fields for reparametrization of the metric in the bosonic string. The Virasoro operators are taken to, for the right moving sector,

$$L_m = \sum (m-n) : b_{m-n} c_n :$$

Show that the Virasoro operators have as their commutators

$$[L_m, L_n] = (m-n) L_{m+n} + \frac{1}{12}\left(-26m^3 + 3m\right)\delta_{m+n,0}.$$

2. The worldsheet fermion is described by creation and annhilation operators b_r^a with anticommutation rules

$$\{b_r^a, b_s^n\} = \eta^{ab}\delta_{m+n,0}.$$

Consider the generators of the superconformal algebra,

$$L_m = \frac{1}{2}\sum (r + \frac{m}{2}) : b_{-r} b_{m+r} :$$

and

$$G_r = \sum \alpha_m b_{r+m}.$$

Show that in the R sector

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{d}{8} m^3\, \delta_{m+n,0},$$

$$[L_m, G_r] = (\frac{m}{2} - r)\, G_{m+r}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{d}{2} r^2\, \delta_{r+s,0}.$$

and in the NS sector that

$$[L_m, L_n] = (m-n)\, L_{m+n} + \frac{d}{8}(m^3 - m)\, \delta_{m+n,0},$$

$$[L_m, G_r] = (\frac{m}{2} - r)\, G_{m+r}$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{d}{2}(r^2 - \frac{1}{4})\, \delta_{r+s,0}.$$

3. Consider the open NN sperstring. Compute the spectrum of physical states up to $\alpha' m^2 = 3$. Show that the degeneracy you find agrees with that predicted by $f_{NS}(w)$ and $f_R(w)$.

4. The low-energy effective action in string frame is given by

$$S = \int d^{26}X \sqrt{-g}\, e^{-2\Phi} \left(R - \frac{1}{12} H_{abc} H^{abc} + 4\partial_a \Phi\, \partial^a \Phi \right) \quad (1)$$

Show that the equations of motions for g_{ab}, B_{ab} and Φ are equivalent to the vanishing of the beta functions

$$\begin{aligned} \beta_{ab}(G) &= R_{ab} + 2\nabla_a \nabla_b \Phi - \frac{1}{4} H_{acd} H_b^{cd} \\ \beta_{ab}(B) &= -\frac{1}{2} \nabla^c H_{cab} + \nabla^c \Phi\, H_{cab} \\ \beta(\Phi) &= -\frac{1}{2} \nabla^2 \Phi + \nabla_a \Phi\, \nabla^a \Phi - \frac{1}{24} H_{abc} H^{abc} \end{aligned}$$

5. Consider the string frame action (1) in d spacetime dimensions. The Einstein frame involves a conformal re-scaling of the metric so that the coefficient of the Ricci scalar is independent of the dilaton. Show that, when written in terms of the Einstein frame metric

$$\tilde{g}_{ab}(X) = e^{-4\tilde{\Phi}/(d-2)}\, g_{ab}(X)$$

the low-energy effective action becomes

$$S = \int d^d X\, \sqrt{-\tilde{g}} \left(\tilde{R} - \frac{1}{12} e^{-8\Phi/d-2} H_{abc} H^{abc} - \frac{4}{D-2} \partial_a \Phi\, \partial^a \Phi \right).$$

It is useful to note that the effect of a conformal rescaling of the metric $g_{ab} \to \Omega^2 g_{ab}$ in d spacetime dimensions on the Ricci scalar is

$$R \to \tilde{R} = \Omega^{-2} R \; - 2(d-1)\Omega^{-1} \nabla_a \nabla^a \Omega^{-1} \; - (n-1)(n-4)\Omega^{-4} \nabla_a \Omega \nabla^a \Omega$$

before attempting this calculation.

6a. The string frame metric produced by N infinite static strings lying in the $(X^0, X^1) \equiv (t, x)$ direction is

$$ds^2 = f(r)^{-1}(-dt^2 + dx^2) + d\vec{X} \cdot d\vec{X}$$

where $\vec{X} = (X_2, \ldots, X_{25})$ labels the space transverse to the string and

$$f(r) = 1 + \frac{g_s^2 N l_s^{22}}{r^{22}}$$

with $r^2 = \vec{X} \cdot \vec{X}$. Consider one further infinite string in this background, lying parallel to the others. Write down the Nambu-Goto action describing the motion of this string. Show that in static gauge $t = R\tau$ and $x = R\sigma$, the low-energy excitations of the string are governed by the effective action,

$$L \approx T \int dtdx \left[-f(r)^{-1} + \frac{1}{2} \left(\frac{d\vec{X}}{dt} \cdot \frac{d\vec{X}}{dt} - \frac{d\vec{X}}{dx} \cdot \frac{d\vec{X}}{dx} \right) + \ldots \right]$$

Interpret this result.

6b. Now include the coupling of the string to a background B-field, which is given by

$$B_{01} = f(r)^{-1} - 1$$

Show that the probe string, suitably oriented and lying parallel to the initial strings, feels no static force.

7. Consider an open string whose ends are constrained to lie on a Dp-brane with a background field strength F_{ab} turned on. Show that the Neumann boundary conditions for the string must be replaced by

$$\partial_\sigma X^a - 2\pi\alpha' \, F^{ab} \, \partial_\tau X_b = 0$$

8a. Show that the Born-Infeld Lagrangian can be written in the form,

$$\mathcal{L}_{BI} \equiv \sqrt{\det(1 + F)} = \exp\left(\frac{1}{4} \text{tr } \ln(1 - F^2) \right)$$

where $F_{ab} = \partial_a A_b - \partial_b A_a$ is the field strength, 1 means the unit matrix, and we have set $2\pi\alpha' = 1$.

8b. Show that the equations of motion arising from the Born-Infeld action are equivalent to the beta function condition for the open string,

$$\beta_a(F) = \left(\frac{1}{1 - F^2} \right)^{bc} \partial_b F_{ca} = 0$$

To do this, it will prove very useful if you can first show the following results:

$$\partial^a \left[\text{tr } \ln(1-F^2)\right] = \left(\frac{F}{1-F^2}\right)^{ab} \partial_c F_{bd} \left(\frac{F}{1-F^2}\right)^{cd}$$

which requires use of the Bianchi identity for F_{ab} and

$$\partial_a \left(\frac{F}{1-F^2}\right)^{ab} = \left(\frac{F}{1-F^2}\right)^{ac} \partial_a F_{cd} \left(\frac{F}{1-F^2}\right)^{db} + \left(\frac{1}{1-F^2}\right)^{ac} \partial_a F_{cd} \left(\frac{1}{1-F^2}\right)^{db}$$