

Supersymmetry: Example Sheet 3

Please *email me* with any comments about these problems, particularly if you spot an error.

1. Consider a theory of a single, free, real scalar field $\phi : \Sigma \rightarrow \mathbb{R}$ with Euclidean action

$$S[\phi] = \frac{1}{2} \int_{\Sigma} h^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi \sqrt{h} d^2x$$

on a Riemann surface Σ . Show that the theory is conformally invariant. Taking Σ to be a 2-torus with flat metric and non-contractible cycles of lengths T and L , show that the torus partition function for this theory is

$$\mathcal{Z} = \frac{V}{2\pi\sqrt{T/L}} q^{-\frac{1}{12}} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)^2},$$

where $q = e^{-2\pi T/L}$ and V is an infra-red cutoff on the target space \mathbb{R} . [You should assume that $\sum_{n=1}^{\infty} n = \zeta(-1) = -\frac{1}{12}$.]

2. Let $H = i\partial/\partial x^0$, $P = -i\partial/\partial x^1$ and $M = x^0\partial/\partial x^1 + x^1\partial/\partial x^0$ be the Hamiltonian, momentum and Lorentz generators of a 1+1 dimensional theory with $\mathcal{N} = (2, 2)$ supersymmetry generated by $Q_{\pm} = \partial/\partial\theta^{\pm} + i\bar{\theta}^{\pm}\partial_{\pm}$ and $\bar{Q}_{\pm} = -\partial/\partial\bar{\theta}^{\pm} - i\theta^{\pm}\partial_{\pm}$. Show that $\{Q_{\pm}, \bar{Q}_{\pm}\} = H \pm P$, and evaluate $[M, Q_{\pm}]$ and $[M, \bar{Q}_{\pm}]$.

In the presence of topological charges, the usual algebra $\{\bar{Q}_{+}, \bar{Q}_{-}\} = 0$ is modified to $\{\bar{Q}_{+}, \bar{Q}_{-}\} = Z$, where Z is a constant. (In particular, Z commutes with all other elements of the superalgebra.) Show that if the theory has a conserved vector $U(1)$ symmetry, then Z must in fact vanish.

3. Consider the following supersymmetric sigma models in $d = 2$:

- (a) A theory of a single chiral superfield X with superpotential $W(X) = aX + b/X$,
- (b) A theory of two chiral superfields X and Y with superpotential $W(X, Y) = gX^2Y + mX^2$,
- (c) A theory of three chiral superfields X , Y and Z with superpotential $W(X, Y, Z) = aY + bX^2Y + cXZ$.

Examine the vacuum moduli space of each of these theories, under the assumption that all fields have canonical kinetic terms. Compute the energy of the vacuum in each case, and find conditions on the coupling constants under which supersymmetry is unbroken.

4. Consider the theory of a complex scalar ϕ and Dirac fermion (ψ_+, ψ_-) with action

$$S[\phi, \psi_{\pm}] = \frac{1}{4\pi} \int \partial^\mu \bar{\phi} \partial_\mu \phi + i\bar{\psi}_- \partial_+ \psi_- + i\bar{\psi}_+ \partial_- \psi_+ \, d^2x,$$

where ϕ describes a map to a target space T^2 , such that $\phi \sim \phi + 2\pi n R_1 + 2\pi i m R_2$, with $(m, n) \in \mathbb{Z} \times \mathbb{Z}$. Show that the theory has a four-dimensional space of ground states, and explain how these states correspond to the de Rham cohomology of T^2 .

5. Consider (0,2) superspace $\mathbb{R}^{2|2}$ with coordinates $(x^\pm, \theta^+, \bar{\theta}^+)$, but no θ^- or $\bar{\theta}^-$. This space has chiral derivatives $D_+ = \partial/\partial\theta^+ - i\bar{\theta}^+ \partial_+$ and $\bar{D}_+ = -\partial/\partial\bar{\theta}^+ + i\theta^+ \partial_+$, and we define (0,2) chiral superfields Φ by the single condition $\bar{D}_+ \Phi = 0$. Write down the most general form of a (0,2)-supersymmetric action. [Recall that the action itself must be bosonic.]
6. A chiral superfield Φ of charge +1 transforms as $\Phi \mapsto e^{i\Lambda} \Phi$ under U(1) gauge transformations, where Λ is again a chiral superfield. Let V be a real superfield transforming under these gauge transformations as $V \mapsto V - i(\Lambda - \bar{\Lambda})$.

- (a) Show that the kinetic term $\int \bar{\Phi} e^V \Phi \, d^4\theta$ is invariant under these gauge transformations.
- (b) Show that Λ can be chosen so that

$$\begin{aligned} V(x, \theta^\pm, \bar{\theta}^\pm) = & \theta^- \bar{\theta}^- A_-(x) + \theta^+ \bar{\theta}^+ A_+(x) - \theta^- \bar{\theta}^+ \sigma(x) - \theta^+ \bar{\theta}^- \bar{\sigma}(x) \\ & + i\theta^- \theta^+ (\bar{\theta}^- \bar{\lambda}_-(x) + \bar{\theta}^+ \bar{\lambda}_+(x)) + i\bar{\theta}^+ \bar{\theta}^- (\theta^- \lambda_-(x) + \theta^+ \lambda_+(x)) \\ & + \theta^- \theta^+ \bar{\theta}^+ \bar{\theta}^- D(x), \end{aligned}$$

where A_\pm define a 1-form, σ is a complex scalar, λ_\pm define a Dirac spinor, and D is a real scalar field.

- (c) Show that the superfield $\Sigma := \bar{D}_+ D_- V$ obeys $\bar{D}_+ \Sigma = 0$, $D_- \Sigma = 0$, and is invariant under gauge transformations.
- (d) Find the component expansion of Σ , and hence show that

$$\int \bar{\Sigma} \Sigma \, d^4\theta \, d^2x = \int \frac{1}{2} F^{\mu\nu} F_{\mu\nu} + i\bar{\lambda}_- \partial_+ \lambda_- + i\bar{\lambda}_+ \partial_- \lambda_+ - \partial^\mu \bar{\sigma} \partial_\mu \sigma + D^2 \, d^2x.$$

Also show that $\int \bar{\Phi} e^V \Phi \, d^4\theta \, d^2x$ generates the minimally coupled gauge-covariant kinetic terms for the scalar field ϕ and fermion ψ in the chiral multiplet Φ .